**On the Maximum Degree
of Minimum Spanning Trees**

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Extended Abstract

Abstract

Motivated by practical VLSI applications, we study the maximum vertex degree in a minimum spanning tree (MST) under arbitrary L_p metrics. We show that the maximum vertex degree in a maximum-degree L_p MST equals the Hadwiger number of the corresponding unit ball. We then determine the maximum vertex degree in a minimum-degree L_p MST; towards this end, we define the MST number, which is closely related to the Hadwiger number. We bound Hadwiger and MST numbers for arbitrary L_p metrics, and focus on the L_1 metric, where little was known. We show that the MST number of a diamond is 4, and that for the octahedron the Hadwiger number is 18 and the MST number is either 13 or 14. We also give an exponential lower bound on the MST number for an L_p unit ball. Implications to L_p minimum spanning trees and related problems are explored.

1 Introduction

The minimum spanning tree (MST) problem has been well-studied, and numerous efficient algorithms exist for MST construction. On the other hand, if we restrict the maximum degree D of an MST, the problem becomes more difficult:

The Bounded Degree Minimum Spanning Tree (BDMST) problem: Given a pointset and an integer D, find a minimum-cost spanning tree with maximum degree $\leq D$.

Finding a BDMST of maximum degree $D = 2$ is equivalent to solving the traveling salesman problem, which is known to be NP-hard [6]. Papadimitriou and Vazirani have shown that that the problem of finding a Euclidean BDMST with $D = 3$ is also NP-hard [14]. On the other hand, they also note that a BDMST with $D = 5$ in the Euclidean plane is actually a Euclidean MST and can therefore be found in polynomial time. The

complexity of the BDMST problem when $D = 4$ remained open. We settle this issue under the rectilinear metric: we show that the rectilinear BDMST problem with $D = 3$ is NP-hard, but that the rectilinear $D = 4$ case is solvable in polynomial time.

The rectilinear metric is common in VLSI applications [15], and has thus been studied by, e.g., Ho, Vijayan, and Wong, who proved that an MST in the rectilinear plane must have maximum degree of $D \leq 8$, and state (without proof) that the maximum degree bound may be improved to $D \leq 6$ [11]. The results of Guibas and Stolfi [9] also imply the $D \leq 8$ bound. In this paper, we show that in the rectilinear plane there always exists an MST with maximum degree of $D \leq 4$ (which is tight). We proceed to analyze the maximum MST degree in three dimensions under the rectilinear metric, where we show a lower bound of 13 and an upper bound of 14 on the maximum MST degree; we also outline an approach to answer the 13 vs. 14 question.

We are generally interested in determining: (i) the maximum degree of a maximum-degree L_p MST, and (ii) the maximum degree of an L_p MST in which the maximum vertex degree is minimized. We show that the maximum degree of a maximum-degree MST under the L_p metric is related to the so-called Hadwiger number of the corresponding L_p unit ball. The relation between MST degree and the packing of convex sets has not been elucidated before, though Day and Edelsbrunner [4] studied the "attractive power" of a point, a topic closely related to the MST degree. For general dimension we give exponential lower bounds on the Hadwiger number and maximum MST degree.

Our study is an outgrowth of the quest for an efficient implementation of the 1-Steiner heuristic [13], which affords a particularly effective approximation to a rectilinear Steiner minimal tree (within 0.5% of optimal for typical input pointsets) [16], and where the central time-consuming loop depends on the maximum MST degree [1] [2]. Our results also have practical utility in new three-dimensional VLSI technologies [10].

The remainder of the paper is as follows. Section 2 establishes the terminology and relates the MST degree to the Hadwiger and MST numbers (the central result is Theorem 4). Section 3 studies the L_1 Hadwiger and MST numbers, and Section 4 considers the Hadwiger and MST numbers for arbitrary L_p metrics. We conclude in Section 5 with open problems.

2 Hadwiger Numbers and MST Numbers

A collection of open convex sets forms a *packing* if no two sets intersect; two sets that share a boundary point in the packing are said to be *neighbors*. The *Hadwiger number $H(B)$* of an open convex set B is the maximum number of neighbors of B considered over all packings of translates of B. There is a vast literature on Hadwiger numbers (e.g., Fejes Tóth [5], Croft *et al.* [3]). Most results address the plane, but there are several results for higher dimensions. In particular, if S is a convex set in $\Re^k$, the Hadwiger number $H(S)$ satisfies

$$k^2 + k \leq H(S) \leq 3^k - 1.$$

It is known that the regular k-simplex realizes the lower bound and the k-hypercube realizes the upper bound [5]. Tighter bounds are known for k-hyperspheres; Wyner [17] showed that the Hadwiger number for spheres is at least $2^{0.207k(1+o(1))}$, and Kabatjansky and Levenštein [12] showed that it is at most $2^{0.401k(1+o(1))}$. Only four Hadwiger numbers for spheres are known exactly; these are the numbers in dimensions 2, 3, 8, and 24. The three-dimensional Hadwiger number has a history dating back to Newton and was only determined much later.

Let $H(k,p)$ be the Hadwiger number for an open L_p unit ball $B(k,p)$ in $\Re^k$. When the unit ball is centered at a point x other than the origin, it is referred to as $B(k,p,x)$. For two points $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ in $\Re^k$, the L_p distance between x and y is $\|xy\|_p = \sqrt[p]{\sum_{i=1}^{k} |x_i - y_i|^p}$. For convenience, if the subscript p is omitted, the rectilinear metric is assumed (i.e., $p = 1$). Let $I(k,p)$ be the maximum number of points that can be placed on the boundary of an L_p unit ball so that each pair of points is at least a unit apart. With respect to a finite set of points $P \subset \Re^k$ and an MST T for P, let $\nu(k,p,P,T)$ be the maximum vertex degree of T. Let $\nu(k,p,P) = \max_{T \in \tau} \nu(k,p,P,T)$, where τ is the set of MSTs for P, and let $\nu(k,p) = \max_{P \subset \Re^k} \nu(k,p,P)$. We need the following result, which is easily established.

Lemma 1 $\|xy\|_p \geq 2$ *if and only if* $B(k,p,x) \cap B(k,p,y) = \emptyset$. *Further,* $\|xy\|_p = 2$ *if and only if* $B(k,p,x)$ *and* $B(k,p,y)$ *are tangent, and a point of tangency lies on the line segment connecting x and y.*

Lemma 2 $H(k,p) = I(k,p)$

Proof: We first show that $I(k,p) \leq H(k,p)$. Suppose that ν points are on the boundary of $B(k,p)$ and are each at least a unit distance apart. Consider placing an L_p ball of radius

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$\frac{1}{2}$ around each point, including one at the origin. By Lemma 1, these balls form a packing, and all the balls touch the ball containing the origin. Therefore, $I(k,p) \leq H(k,p)$.

We now show that $H(k,p) \leq I(k,p)$. Consider a packing of L_p unit balls, and choose one to be centered at the origin. Draw an edge from the origin to the center of each neighboring ball; we claim that the intersection of these edges with $B(k,p)$ forms a pointset where each pair of points is separated by at least a unit distance. If this is not the case, one of these points would be contained within one ball and tangent to another, a contradiction. ◻

Lemma 3 $\nu(k,p) = H(k,p)$

Proof: We show that $\nu(k,p) \leq I(k,p)$. Let x be a point, and let $y_1, \ldots, y_\nu$ be points adjacent to x in an MST, indexed in order of increasing distance from x. Note that (y_i, y_j) must be a longest edge in the triangle (x, y_i, y_j), and further note that the MST restricted to x and $y_1, \ldots, y_\nu$ is a star centered at x. Draw a small L_p ball around x, without loss of generality a unit ball, and consider the intersection of the segments (x, y_i), $1 \leq i \leq \nu$, with $B(k, p, x)$. Let these intersection points be called $\hat{y}_i$, $1 \leq i \leq \nu$, and suppose there is a pair $\hat{y}_i$ and $\hat{y}_j$, $i < j$, with $\|\hat{y}_i \hat{y}_j\|_p \leq 1$. Note that $(\hat{y}_i, \hat{y}_j)$ is the shortest edge on the triangle $(x, \hat{y}_i, \hat{y}_j)$, and $(x, \hat{y}_j)$ is a longest edge. Now consider similar triangle (x, y_i, z), where z is a point on the edge (x, y_j). The path from y_j to z to y_i is shorter than the length of (x, y_j), so (y_i, y_j) is not a shortest edge in triangle (x, y_i, y_j), a contradiction. We note this bound is tight for pointsets that realize $I(k,p)$. ◻

We now consider a slightly different number $\hat{\nu}(k,p)$, which is closely related to $\nu(k,p)$. Recall the definition of $\nu(k, p, P, t)$. Let $\hat{\nu}(k, p, P) = \min_{T \in \tau} \nu(k, p, P, T)$, where τ is the set of MSTs for P, and let $\hat{\nu}(k,p) = \max_{P \subset \Re^k} \nu(k, p, P)$.

Although it is clear that $\hat{\nu}(k,p) \leq \nu(k,p)$, it is not clear when this inequality is strict. To count $\hat{\nu}(k,p)$, we define the MST number $M(k,p)$ and the number $\hat{I}(k,p)$. The number $M(k,p)$ is defined similarly to the Hadwiger number $H(k,p)$, except that the translates of the L_p unit ball $B(k,p)$ are slightly magnified. The underlying packing consists of $B(k,p)$ and multiple translated copies of $(1 + \epsilon)B(k,p)$; $M(k,p)$ is the supremum over all $\epsilon > 0$ of the maximum number of neighbors of $B(k,p)$ over all such packings. It is clear that $M(k,p) \leq H(k,p)$. The number $\hat{I}(k,p)$ is the number of points that can be placed on boundary of $B(k,p)$ so that each pair is strictly greater than one unit apart.

Consider a set $S = \{x_1, \ldots, x_n\}$ of n points in $\Re^k$, and suppose their interdistances are measured in an L_p metric. For convenience, let $N = 2^{\binom{n}{2}}$. Let $S_1, \ldots, S_N$ be the set of

sums of the interdistances, one sum for each distinct subset. Let

$$0 < \delta = \min_{1 \le i < j \le N} \{ |S_i - S_j| : |S_i - S_j| > 0 \}$$

A *perturbation* of a pointset S is a bijection to a second set $S' = \{x'_1, \ldots, x'_n\}$ (for convenience, suppose the indices indicate the bijection); we say that a perturbation of S is *small* if

$$\sum_{i=1}^{n} \|x_i x'_i\|_p < \frac{1}{2}\delta$$

In dealing with spanning trees of S and perturbation S', we assume that the vertex set $[n]$ consists of the integers 1 to n, where vertex i corresponds to point x_i or point x'_i. The *topology* of a tree over vertex set $[n]$ is the set of edges in the tree.

Theorem 4 *Let S be a set of points, and let S' be a set of points corresponding to a small perturbation of S. Then the topology of an MST for S' is also a topology for an MST for S.*

Proof: Let T be an MST for S, and let T' be an MST for S'. Let $l(T)$ and $l(T')$ be the lengths of T and T', respectively. Then $l(T) - \frac{1}{2}\delta < l(T') < l(T) + \frac{1}{2}\delta$. Consider the tree $\hat{T}$ with the same topology as T' but with respect to pointset S. Now, $l(T') - \frac{1}{2}\delta < l(\hat{T}) < l(T') + \frac{1}{2}\delta$, so $l(T) - \delta < l(\hat{T}) < l(T) + \delta$. Recall that δ is defined to be the minimum positive difference between the sums of two distinct subsets of interdistances. This implies that $l(T) = l(\hat{T})$, so $\hat{T}$ is also an MST for S. $\square$

Lemma 5 $\hat{\nu}(k, p) = M(k, p)$

Proof: Let S be a set of points, and let δ be defined as above. Place a small L_p ball about each point $x \in S$ (without loss of generality a unit ball, though the intent is that x is the only point inside $B(k, p, x)$), and connect each distinct pair (x, y), $x, y \in S$, with a line segment. Consider the intersections of these edges with $B(k, p, x)$. Perform a small perturbation on S so that no two intersection points have length 1. Repeat the argument used in the proof of Lemmas 2 and 3, this time with balls of the form $(1+\epsilon)B(k, p)$, for small $\epsilon > 0$. The first part shows that $\hat{I}(k, p) = M(k, p)$, and the second that $\hat{\nu}(k, p) \le \hat{I}(k, p)$. This bound is tight for pointsets that realize $\hat{I}(k, p)$. $\square$

3 The L_1 Metric

Hadwiger numbers are notoriously difficult to compute. In this section, we determine the 2 and 3 dimensional Hadwiger numbers for the diamond and octahedron, respectively. The first of these numbers is well-known, but we could not find any reference for the octahedron. We also study the MST numbers, obtaining a value of 4 in 2 dimensions, and obtaining bounds in higher dimensions.

Lemma 6 $\nu(2,1) = 8$

Proof: We show that $I(2,1) = 8$. Two diagonal lines through a point p partition the $\Re^2 - \{p\}$ into 8 disjoint regions, four of dimension 2, and four of dimension 1 (Figure 1(a)). We now show that for any two distinct points u and w that lie in the same region, either $\|wu\| < \|wp\|$ or else $\|uw\| < \|up\|$. This is obvious for the 1 dimensional regions. Consider u, w in one of the 2 dimensional regions (Figure 1(a)). Assume wlog that $\|up\| \leq \|wp\|$ (otherwise swap u and v). Consider the diamond D with corner p, center c, and u on its boundary (Figure 1(b)). Let a ray starting at p and passing through w intersect D at b. By the triangle inequality, $\|wu\| \leq \|wb\| + \|bu\| < \|wb\| + \|bc\| + \|cu\| = \|wb\| + \|bc\| + \|cp\| = \|wp\|$. Thus every one of the 8 regions can contain at most 1 point. The pointset $\{(0,0), (\pm 1, 0), (0, \pm 1), (\pm\frac{1}{2}, \pm\frac{1}{2})\}$ shows that this bound is tight. ∎
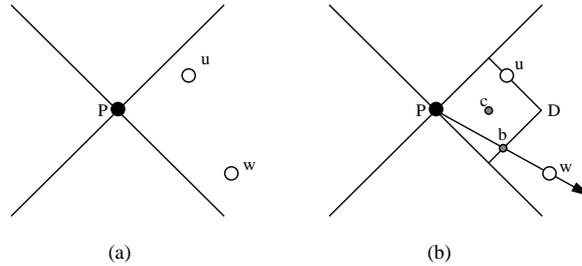

Figure 1: A partition of $\Re - \{p\}$ into 8 disjoint regions such that for any two points u and w that lie in the same region (a), either $\|wu\| < \|wp\|$ or else $\|uw\| < \|up\|$ (b).

Theorem 7 $\nu(3,1) = 18$

Proof: Again, we consider $I(3,1)$. Consider an *cuboctahedral* partition of $\Re^3$ into 14 disjoint regions corresponding to the faces of a truncated cube (Figure 2(a-b)), i.e., 6

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congruent pyramids with *square* cross-section (Figure 2(c)) and 8 congruent pyramids with *triangular* cross-section (Figure 2(d)). Most of the region boundaries are included into the triangular pyramid regions as shown in Figure 3(c), with the remaining boundaries forming 4 new regions (Figure 3(d)), to a total of 18 regions. Using analogous arguments to those in the proof of Lemma 6, it can be shown that for any two points u and w lying in one of these regions, either $\|wu\| < \|wp\|$ or else $\|uw\| < \|up\|$. The pointset $\{(0, 0, 0), (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (\pm\frac{1}{2}, \pm\frac{1}{2}, 0), (0, \pm\frac{1}{2}, \pm\frac{1}{2}), (\pm\frac{1}{2}, 0, \pm\frac{1}{2})\}$ shows that this bound is tight. $\square$
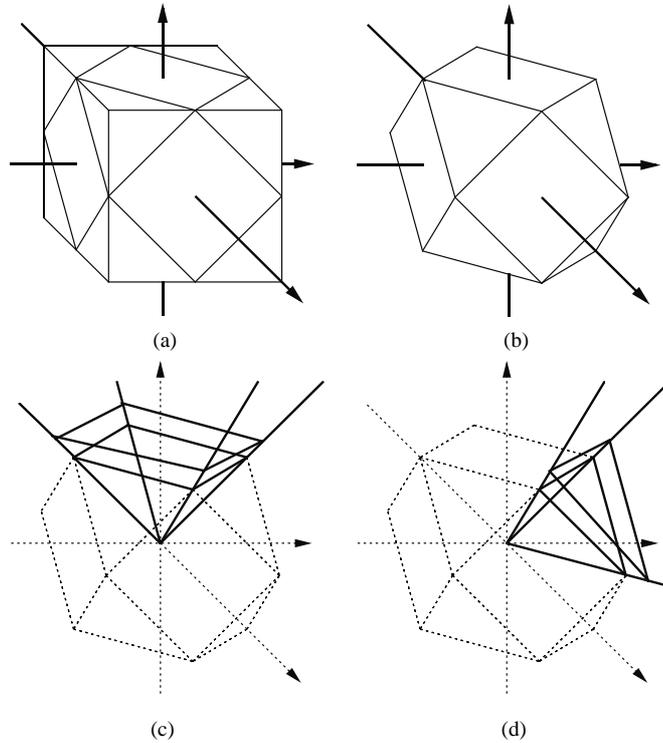

Figure 2: A truncated cube (a-b) induces a 3-dimensional cuboctahedral partition of space into 14 regions: 6 square pyramids (c), and 8 triangular pyramids (d).

Theorem 8 $\hat{\nu}(2, 1) = 4$

Proof: The pointset $\{(0, 0), (\pm 1, 0), (0, \pm 1)\}$ establishes a lower bound of 4. To get the upper bound of 4, consider $\hat{I}(2, 1)$. Partition the plane into 8 disjoint regions, as in the proof of Lemma 6; at most one point can be in the closure of each of the four 2-dimensional regions, proving the result. $\square$
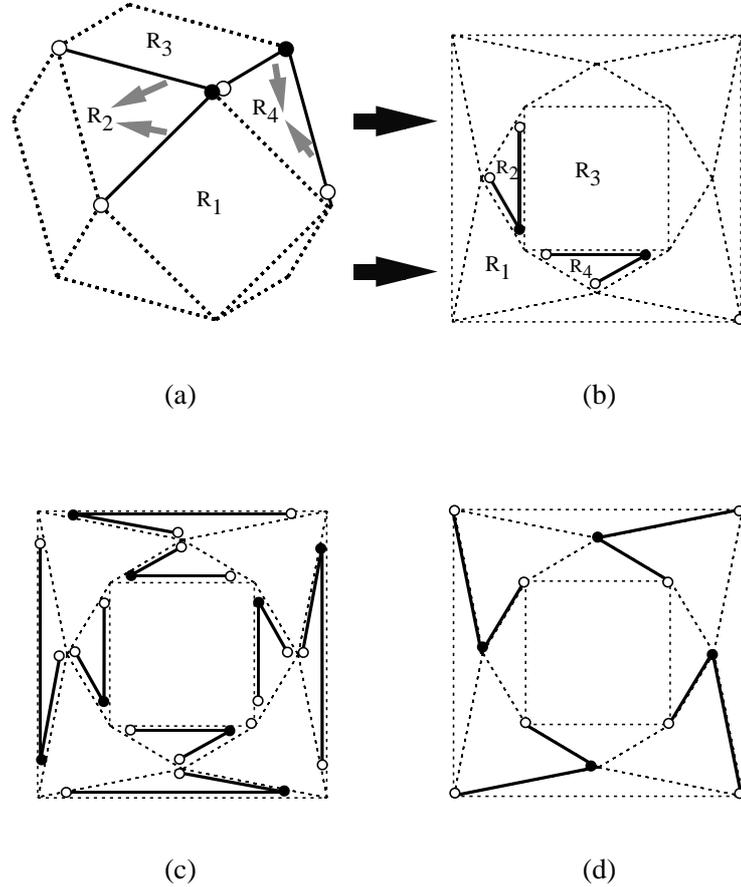
Figure 3: Assigning the boundary points to the various region: an open dot indicates an open interval, while a solid dot indicates a closed interval. A topological mapping of the cuboctahedron (a) is shown in (b). The various boundaries are included into the triangular pyramid regions (c), while the square pyramids do not contain any boundary points. The remaining boundaries form 4 new regions (d), bringing the total to 18.

Theorem 8 has an interesting consequence on the complexity of the BDMST problem restricted to the rectilinear plane:

Instance: A planar pointset $P = \{x_1, \ldots, x_n\}$, and integers D and C.

Question: Is there a rectilinear spanning tree with maximum degree $\leq D$ and cost $\leq C$?

If $D = 4$, the question can be decided in polynomial time. On the other hand, if $D = 2$, the problem is essentially a rectilinear traveling salesman problem (a *wandering* salesman problem, since the tour is a path rather than a circuit), and it is NP-complete. It turns out

that the $D = 3$ question is also NP-complete, using a proof identical to the one appearing in Papadimitriou and Vazirani [14]. Their result is for the corresponding Euclidean problem, but they restrict all their points to being on a special type of *grid graph*, and the proof holds in the rectilinear metric as well. We can summarize the rectilinear and Euclidean results in the following table and note the interesting contrast for $D = 4$:

D	Euclidean	rectilinear
2	NP-complete	NP-complete
3	NP-complete	NP-complete
4	open	polynomial
≥ 5	polynomial	polynomial

Theorem 9 $13 \leq \hat{\nu}(3,1) \leq 14$

Proof: For the lower bound, the following pointset shows that $\hat{I}(k,p) \geq 13$: $\{(0,0,0),$ $(\pm 100, 0, 0), (0, \pm 100, 0), (0, 0, \pm 100), (47, -4, 49), (-6, -49, 45), (-49, 8, 43), (-4, 47, -49),$ $(-49, -6, -45), (8, -49, -43), (49, 49, 2)\}$. The upper bound follows from the observation that any two points lying in the closure of one of the 14 main regions of the cuboctahedral partition (see the proof of Theorem 7 and Figure 2) must be within distance 1 of each other. $\qquad\square$

We note that there is an elementary means to settle the 13 vs. 14 question raised in Theorem 9. Suppose we are trying to decide whether 14 points can be placed on the surface of a unit octahedron so that each pair is greater than a unit distance apart. The relationship between point (x_i, y_i, z_i) and point (x_j, y_j, z_j) can be phrased as the following inequality:

$$|x_i - x_j| + |y_i - y_j| + |z_i - z_j| > 1$$

This is subject to the constraint that $|x_i| + |y_i| + |z_i| = 1$ and $|x_j| + |y_j| + |z_j| = 1$. The absolute values can be removed if the relative order between x_1 and x_2, etc., is known. We can therefore consider all permutations of the coordinates of the 14 points and produce the corresponding inequalities. If the inequalities corresponding to a particular permutation are simultaneously satisfied, $\hat{\nu}(3,1) = 14$, otherwise $\hat{\nu}(3,1) = 13$. Feasibility can be settled by determining whether a particular polytope contains a nonempty relative interior (this approach is easily extended to any dimension k). We have not managed to settle the 13 vs. 14 question manually, and the above procedure seems impractical due to the large number of resulting inequalities.

We now deal with the Hadwiger and MST Numbers for the k-crosspolytope. The Hadwiger number satisfies the following lower bound.

Theorem 10 $\hat{\nu}(k,1) = \Omega(2^{0.0312k})$

Proof: Consider the family $F(j)$ of points $(\pm\frac{1}{j},\ldots,\pm\frac{1}{j},0,\ldots,0)$, where j is an integer between 1 and k. (Here, the j nonzero terms can be arbitrarily interspersed in the vector.)

Each member of $F(j)$ is distance 1 from the origin; the distance between $x \in F(j)$ and $y \in F(j)$ depends on the positions and signs of the nonzero terms. Given $x = (x_1, x_2, \ldots, x_k) \in F(j)$, let $\bar{x}$ be the binary vector containing a 1 in bit i if $x_i \neq 0$ and a 0 in bit i if $x_i = 0$. If the Hamming distance between $\bar{x}$ and $\bar{y}$ is at least j, then $\|xy\| \geq 1$. (The Hamming distance between bit vectors a and b is the number of bit positions in which a and b differ.) We want to find a large set of $\bar{x}$ that are mutually Hamming distance greater than j apart.

Consider the set $V(j)$ of bit vectors containing exactly j 1's; $|V(j)| = \binom{k}{j}$. Form a graph $G(t) = (V(t), E(t))$ for which $(\bar{x}, \bar{y}) \in E(t)$ if and only if the Hamming distance between $\bar{x}$ and $\bar{y}$ is at most j. Note that $G(t)$ is regular with degree $d(j) = \sum_{i=1}^{\lfloor j/2 \rfloor} \binom{j}{i}\binom{k-j}{i}$. To see this, we determine the number of edges adjacent to $\bar{x} = (1,\ldots,1,0,\ldots,0)$, where there are j 1's. The set of vectors in $V(j)$ adjacent to $\bar{x}$ can be partitioned into vectors that contain i 0's in the first j positions, $1 \leq i \leq \lfloor \frac{j}{2} \rfloor + 1$. For a given i, there are $\binom{j}{i}$ ways to choose the 0 positions and $\binom{k-j}{i}$ positions to place the displaced 1's in the last $k - j$ positions.

Here is our strategy to find a subset of $V(j)$ of large cardinality that are mutually far apart: choose a vertex, delete its neighbors, and continue. The number of vertices chosen must exceed $|V(j)|/d(j)$. Suppose that $cj = 16\sqrt{e}j = k$. Then

$$\frac{|V(j)|}{d(j)} \geq \frac{\binom{k}{j}}{(\frac{j}{2}+1)\binom{j}{\frac{j}{2}}\binom{k-j}{\frac{j}{2}}} \geq \frac{\binom{k}{j}}{(\frac{j}{2}+1)\frac{4^j}{\sqrt{\pi j}}\binom{k-j}{\frac{j}{2}}} > c'\frac{c^{j/2}}{4^j\sqrt{j}} = c''\left(\frac{c}{16}\right)^{\frac{k}{2c}}\frac{1}{\sqrt{k}}$$

Here, c' and c'' are constants; the approximation to $\binom{j}{\frac{j}{2}}$ is from Graham *et al.* [7]. Substituting for c gives the result. $\qquad\qquad\square$

4 Arbitrary L_p Metrics

In this section, we provide bounds on $M(k,p)$ for general L_p metrics.

Theorem 11 $\hat{\nu}(k,p) = \Omega(\sqrt{k}2^{n(1-E(\alpha))})$, where $\alpha = \frac{1}{2^p}$ and $E(x) = x\lg\frac{1}{x} + (1-x)\lg\frac{1}{1-x}$

Proof: Consider the vertices of the k-hypercube $(\pm1,\ldots,\pm1)$. Each of these points is $k^{1/p}$ from the origin. On the other hand, if points x and y differ from each other in j positions,

they are distance $j2^{1/p}$ from each other. If $j2^{1/p} > k^{1/p}$, then x and y are further from each other than they are from the origin.

We need to find the largest cardinality set of points on the k-hypercube that differ in at least $J = \frac{k}{2^p}$ positions. To do this, construct a graph G whose vertex set is the set of binary strings of length k, and for which there is an edge between string a and string b if and only if the Hamming distance between a and b is at most J. Proceed in the same manner as in the proof of Theorem 10, except that $d(J) = \sum_{i=1}^{J} \binom{k}{i}$. The number of vertices chosen must exceed $2^k/d(J)$. Now,

$$\sum_{i \leq \alpha k} \binom{k}{i} = 2^{kE(\alpha) - \frac{1}{2} \lg k + O(1)}$$

for $0 < \alpha < \frac{1}{2}$ (see Graham et $al.$ [7], Chapter 9, Problem 42). Note that $\alpha = \frac{1}{2^p}$, so

$$\frac{2^k}{d(J)} = \sqrt{k} 2^{k(1 - E(2^{-p}))}$$

$\square$

Theorem 11 shows that for any fixed $p > 1$, $\hat{\nu}(k, p)$ grows exponentially in the dimension. Note that this bound is less than the bound obtained by Wyner (for $H(k, 2)$, it is $\Omega(2^{0.189k})$ since $E(\frac{1}{4}) = \frac{3}{4} \lg 3 - 1 \approx 0.189$.), but it is sufficient for our purposes.

It is well known that $H(k, p) \leq 3^k - 1$. (See Fejes Tóth for these results.) In 2-dimensional space, the Hadwiger number is largest for L_1 and L_∞, the only planar L_p metrics with Hadwiger number 8. For all other L_p metrics, the Hadwiger number is 6. On the other hand, the planar MST number is smallest for L_1 and L_∞, having a value of 4, and it is easily seen to be 5 for all other L_p metrics.

These observations raise an interesting question regarding MST numbers in $\Re^k$: how does the MST number behave as a function of p? Note that the maximum Hadwiger number is achieved by parallelotopes. In the next sequence of results, we derive the MST number for the L_∞ unit ball (i.e., the k-hypercube). We then show that the MST number is not maximized in the L_∞ metric in any dimension. (The 2- and 3-dimensional bounds were already established.)

Theorem 12 $\hat{\nu}(k, \infty) = 2^k$

Proof: We first show the result for $p = \infty$; note that the L_p unit ball is a k-hypercube. The upper bound is established by considering $\hat{I}(k, p)$ and noting that at most one point

can be placed in each k-ant (the k-dimensional analogue of quadrant / octant). The lower bound is established by considering the set of 2^k vertices of a k-hypercube. $\qquad\square$

Theorem 13 *For each k, there is a p such that $\hat{\nu}(k,p) > 2^k$.*

Proof: Consider the pointset $(-1, \pm1, \ldots, \pm1)$, $(\epsilon, \pm\delta, \ldots, \pm\delta)$, and $(k^{1/p}, 0, \ldots, 0)$, where $(\epsilon^p + (k-1)\delta^p) = k$. It is possible to choose ϵ, δ, and p so that each pair of points is on the surface of a L_p ball of radius $k^{1/p}$, and each interdistance is greater than $k^{1/p}$. $\qquad\square$

5   Conclusion

We studied the maximum vertex degree in an L_p MST. We showed that the maximum vertex degree in a maximum-degree L_p MST equals the Hadwiger number of the corresponding unit ball. We then determined the maximum vertex degree in a minimum-degree L_p MST; to this end, we defined a second number, the MST number, which is closely related to the Hadwiger number. Motivated by a practical application, we calculated or bound Hadwiger and MST numbers for L_p metrics, concentrating on the L_1 metric, where little is known. We showed that the MST number of a diamond is 4, that the Hadwiger and MST numbers of an octahedron are 18 and either 13 or 14, respectively, and gave an exponential lower bound on the MST number of a k-crosspolytope. We also showed that the MST number for an L_p unit ball, $p > 1$, is exponential in the dimension. Implications to L_p minimum spanning trees and related problems were explored. Remaining open problems include:

1. Whether the MST number for L_1 in three dimensions 13 or 14;

2. The complexity of computing a planar Euclidean MST with maximum degree 4;

3. Tighter bounds on the Hadwiger and MST numbers for arbitrary k and p.

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