Uniform Antimatroid Closure Spaces*

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Abstract

Often the structure of discrete sets can be described in terms of a closure operator. When each closed set has a unique minimal generating set (as in convex geometries in which the extreme points of a convex set generate the closed set), we have an antimatroid closure space. In this paper, we show there exist antimatroid closure spaces of any size, of which convex geometries are only a sub-family, all of whose closed sets are generated by precisely the same number of points. We call them *uniform* closure spaces.

The issue of whether a planar convex geometry, that is a discrete set of points in in the plane, exists in which all convex configurations are triangles, with no quadrilaterals; or as quadrilaterals and triangles, with no pentagons; *etc.*, has fascinated combinatorial mathematicians for many years. Our results throw light on these kinds of questions, even though they do not resolve them.

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1 Introduction

By a discrete space we mean a set of elements, points, or other phenomena which we will generically call our *universe*, denoted by $\mathbf{U}$. Individual points of $\mathbf{U}$ will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. By $2^{\mathbf{U}}$, we mean the powerset on $\mathbf{U}$, or collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters:

We say $(\mathbf{U}, \varphi)$ is a **closure space** if φ is a closure operator satisfying the three standard closure axioms:

$Y \subseteq Y.\varphi$
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$
$Y.\varphi.\varphi = Y.\varphi^2 = Y.\varphi$

It is an **antimatroid** closure space if, in addition, it satisfies

$X.\varphi = Y.\varphi$ implies $(X \cap Y).\varphi = X.\varphi = Y.\varphi$.

(From now one, we will use *closure space* to mean an antimatroid closure space.) The last axiom is non-standard. It is not hard to show that closure operators which satisfy this additional axiom are *uniquely generated* in the sense that for any set Y, there exists a unique minimal set $X \subseteq Y$ such that $X.\varphi = Y.\varphi$. One can also show [14] that

Theorem 1.1 *A closure operator is uniquely generated if and only if it satisfies the anti-exchange property*

if $p, q \notin Y.\varphi$ *then* $p \in (Y \cup \{q\}).\varphi$ *implies* $q \notin (Y \cup \{p\}).\varphi$.

In contrast, any set of elements $\mathbf{U}$ with an operator σ satisfying the first three closure axioms, together with the Steinitz-MacLane exchange axiom

if $p, q \notin Y.\sigma$ then $p \in (Y \cup \{q\}).\sigma$ implies $q \in (Y \cup \{p\}).\sigma$

is called a *matroid* [9] [15] [1].[1] Because $(\mathbf{U}, \varphi)$ satisfies the anti-exchange axiom, the adjective *antimatroid* is completely descriptive [2] [8].[2] Other common names for this concept are *APS greedoid, shelling structure* [7], *alignment* [6], or *convex geometry* [4] provided only that one further requires the empty set, $\emptyset$, to be closed. By the **generator** of Y, or basis[3] of Y, denoted $Y.\beta$, we mean a minimal set X such that $X.\varphi = Y.\varphi$. Because φ is uniquely generated, $Y.\beta$ is uniquely defined. In convex geometries, the generators are called *extreme* points, which is quite descriptive [4].

[1] The closure operator σ of a matroid is normally called the spanning operator.

[2] Antimatroid closure spaces are far more abundant than one might expect. For example, there exist at least 202 distinct closure spaces comprised of 5 elements. More generally, it can be shown that there exist more than n^n distinct, non-isomorphic closure spaces provided $n \geq 10$ [12]. Similarly, there are many different closure operators, φ.

[3] The term "basis" has so many connotations, especially with respect to vector spaces and their change of basis, that we prefer the more neutral "generator".

Antimatroid closure spaces have been studied in [14], in which the subsets $X, Y \subseteq \mathbf{U}$ are partially ordered by $\leq_\varphi$, where

$$X \quad \leq_\varphi \quad Y \qquad \text{if and only if} \qquad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi \tag{1}$$

This is a partial order on *all* the subsets of $\mathbf{U}$, not just its closed subsets. It is possible to show that this partial ordering of $2^{\mathbf{U}}$ is, in fact, a well structured lattice, $\mathcal{L}$, called the **closure lattice** of $(\mathbf{U}, \varphi)$. A representative closure operator over a partially ordered set, $\mathbf{U}$, is φ_C, in which the closure of a set Y is $Y.\varphi = \{y \mid \exists x, z \in Y \quad \text{such that} \quad x \leq y \leq z\}$. In this case, the minimal and maximal points of Y constitute its generators. Figure 1(a) is a typical poset on 6 points; Figure 1(b) illustrates its closure lattice, $\mathcal{L}_{(\mathbf{U},\varphi_C)}$. The regularity

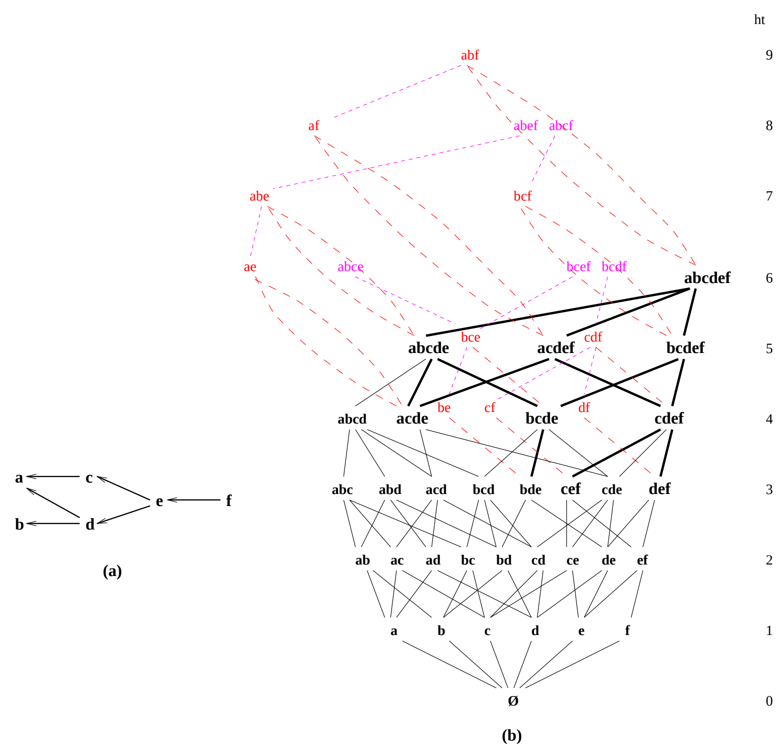

Figure 1: A Closure Lattice, $\mathcal{L}_{(\mathbf{U},\varphi_C)}$

of structure suggested by this figure really exists, *c.f.* [14]. The collection of **closed** subsets, for which $Y = Y.\varphi$, forms a sublattice $[\emptyset, \mathbf{abcdef}]$, denoted in this figure by bolder strings and joined by solid lines that are generally inclined from the lower left to the upper right which denote covering relationships. This sublattice of closed subsets is lower semimodular;

that is, if Z covers Y_1 and Y_2, then Y_1 and Y_2 must cover a common closed set X.[4] Edelman [3] has pushed this further and demonstrated that every antimatroid is meet distributive.

Generators are connected to the corresponding closed sets that they generate by dashed lines generally inclined from lower right to the upper left. It can be shown that each of the lattice intervals $[Y.\varphi, Y.\beta]$ is a boolean lattice. In the case of the 5 closed sets **abcdef**, **abcde**, **acdef**, **bcdef** and **acde**, whose generators are *abf*, *abe*, *af*, *bcf* and *ae* respectively, we indicate these boolean intervals with a dashed elliptical outline. The **height** of any element Y, denoted $ht(Y)$, is indicated at the right.

A few constituent sets have also been shown. The dotted lines denote a few of the covering relationships between non-closed elements in different boolean intervals. These covering relationships, which we denote by $X \prec_\varphi Y$, do indeed echo those of the closed subgraph sublattice. In particular, we have the following results from [14].

Theorem 1.2 (Fundamental Covering Theorem) *If $p \notin X$ then*

(a) $X \leq_\varphi X \cup \{p\}$ *if and only if* $p \notin X.\varphi$
(b) $X \cup \{p\} \leq_\varphi X$ *if and only if* $p \in X.\varphi$

where (a) is a cover if and only if $(X \cup \{p\}).\varphi = X.\varphi \cup \{p\}$ *and*
(b) is always a covering relationship.

Moreover, if φ is uniquely generated then (a) and (b) characterize all covering relations in $(2^{\mathbf{U}}, \leq_\varphi)$.

Lemma 1.3 *If φ is uniquely generated, and if $Z \neq \emptyset$ is closed,*

(a) $p \in Z.\beta$ *if and only if* $Z - \{p\}$ *is closed,*
in which case $Z.\beta - \{p\} \subseteq (Z - \{p\}).\beta$;
(b) $p, q \in Z.\beta$ *implies there exist closed sets* $Y_p, Y_q \subset Z.\varphi$
such that $p \in Y_p, q \in Y_q$ *and* $p \notin Y_q, q \notin Y_p$;
(c) if $\emptyset.\varphi = \emptyset$, *there exists* $p \in Z.\varphi$ *such that* $\{p\}$ *is closed.*

Let $Z.\!\downarrow$ denote the set $\{Y_1, Y_2, ...Y_n\}$ of sets covered Z in $\mathcal{L}$. If Z is closed, then because of Lemma 1.3(a), $Z.\!\downarrow$ is uniquely determined by $Z.\beta$, and conversely. In particular, $|Z.\beta| = |Z.\!\downarrow|$.

[4] The lower semimodularity of closed subsets partially ordered by inclusion has been repeatedly discovered by many authors. See Monjardet [10] for an interesting summary.

Theorem 1.4 (Fundamental Structure Theorem) *Let $X.\varphi \le_\varphi Y.\varphi$ and let $X \in [X.\varphi, X.\beta]$. There exists a unique $Y \in [Y.\varphi, Y.\beta]$ such that $X \le_\varphi Y$, where Y is minimal wrt. $\le_\varphi$ (maximal wrt. $\subseteq$). Moreover $Y = X \cup \Delta$ where $\Delta = Y.\varphi - X.\varphi$ and $Y = Y.\varphi - \delta$ where $\delta = X.\varphi - X$.*

Figure 1(b) illustrates this theorem. Every interval $[Y.\varphi, Y.\beta]$ can be projected "upwards". By Theorem 1.2, every covering relation is marked by the difference of just one element between the two sets. Consequently, it can be illustrative to label covering relations (edges) with the corresponding element.

2 Uniform Closure Spaces

A closure space $(\mathbf{U}, \varphi)$ is said to be **atomic** if every singleton set, *i.e.* point, $\{x\} \in 2^{\mathbf{U}}$ is closed.[5] Figure 1(b) is atomic. These singleton sets are collectively called the **atoms** of $\mathcal{L}_{(U,\varphi)}$ and denoted by the set A. Readily, $\emptyset$ is closed; and $a \in A$ if and only if a covers $\emptyset$. (Note that we normally represent points, or atoms, in $\mathbf{U}$ with lowercase letters and sets of $\mathbf{U}$ with uppercase. In the case of atoms, it is just too convenient to let the single lower case letter denote both the atom and the singleton set as a whole. Frequently we let a string of lower case letters denote the set comprised of those atoms.)

Let A_Y, sometimes called the **atom set** of Y, denote the set of atoms a in $\mathcal{L}_{(U,\varphi)}$ such $a \le_\varphi Y$. It is not hard to show that the sublattice of closed subsets is also atomic in the sense that

$$Y \quad \text{closed implies} \qquad Y = \bigvee_{a \in A} \{a \le_\varphi Y\} = \bigvee_{a \in A_Y} a \tag{2}$$

even though $\mathcal{L}_{(U,\varphi)}$ as a whole cannot satisfy (2). Further, for all closed sets Y, $|Y| = |A_Y| = ht(Y)$ in $\mathcal{L}_{(U,\varphi)}$. That is, in atomic closure spaces, the cardinality of a closed set Y, regarded as a set of points in $\mathbf{U}$, equals the cardinality of its atom set in $\mathcal{L}_{(U,\varphi)}$, which in turn equals its height in $\mathcal{L}_{(U,\varphi)}$. This facilitates reasoning which transfers the focus form sets of points in a closure space to elements of a lattice and *vice versa*.

An atomic closure space is said to have **closure dimension** $d \ge 1$ if (a) every subset $Y \subseteq \mathbf{U}$ with $|Y| < d$ is closed, implying (b) for all closed sets $Y, |Y| \ge d$, implies $|Y.\beta| \ge d$.[6] The interval closure operator, φ_C on posets, such as shown in Figure 1(b), has closure dimension = 2. In [11] it is called a convex closure. Chordal graphs under monophonic closure [6] also have closure dimension = 2. A planar convex geometry has closure dimension

[5] Not all closure spaces are atomic. In particular, the left and right ideal closures, φ_L and φ_R on a poset are not atomic. See [14]

[6] A corollary to the results in [12] demonstrates that a finite closure space of dimension 0, is impossible.

3, provided no three points are colinear. If no 4 points are coplanar, a convex geometry in 3-space has closure dimension 4. For these geometric spaces, the antimatroid closure dimension is one greater than its matroid dimension.

If the generator of a closed set Y is the set itself, *i.e.* $Y.\beta = Y.\varphi$, we call Y a **simple** closed set. So a generating set $Y.\beta$ is **non-simple** if $Y.\beta \subset Y.\varphi$. Readily, if $(\mathbf{U}, \varphi)$ is atomic, with dimension d, then for all non-simple generator sets $|Y.\beta| \geq d$. An atomic closure space of dimension d in which every non-simple set of generators has cardinality precisely d is said to be **uniform**. That is, for all closed sets Y, if $|Y| \geq d$ then $|Y.\beta| = d$.

Figure 2 illustrates a uniform closure space on 5 points for $|Y.\beta| = 2$. Readily each closed

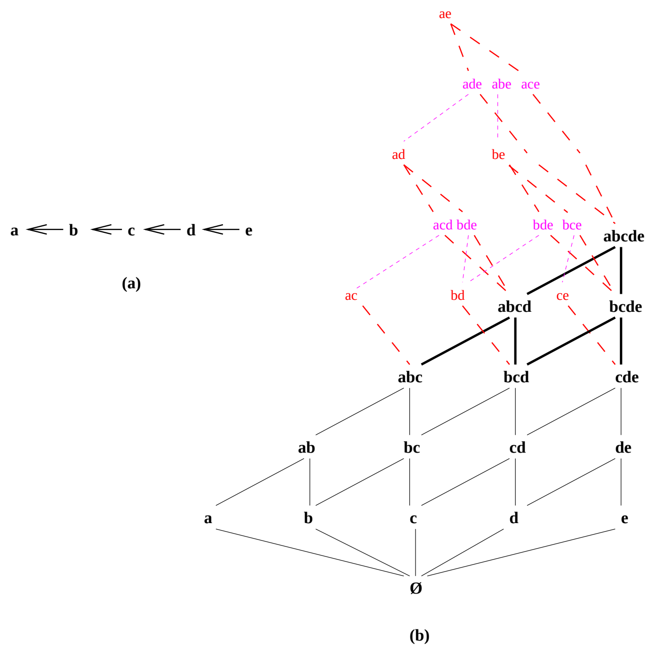

Figure 2: A uniform closure space, d=2

set corresponds to a subinterval of this totally ordered set. And it is equally clear that for all n, we have a similar uniform closure space of dimension $d = 2$. Figure 3 illustrates another closure space of dimension 2 that is not uniform; but it is nearly so. We say a closure space is **weakly uniform** if for all non-simple closed sets Y, $|Y| \geq d$ implies $|Y.\beta| = d$. Observe that $|\{abc\}.\beta| = 3 > 2$, but $\{abc\}$ is a simple closed set. Given a uniform closure space, it is not hard to derive many weakly uniform closure spaces of the same dimension.

While the closure space of Figure 2 can be derived from the intersections of sub-intervals of a total order, Figure 3 has no such equivalent characterization. Many closure spaces appear to be "unrealizable" in terms of other more familiar closure operators. The issue

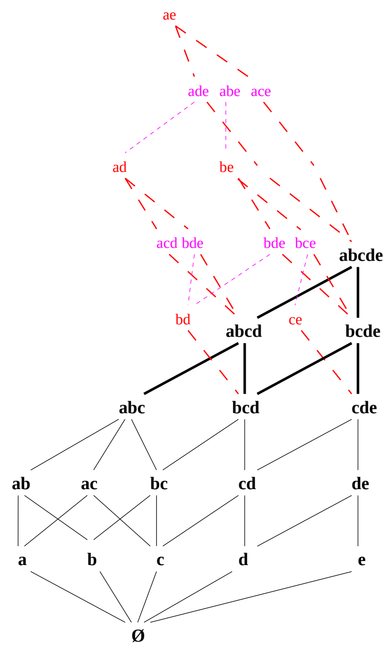

Figure 3: A weakly uniform closure space, d=2

as to whether specific operators, such as convex geometric closure, define characteristic families of closure lattices remains an interesting open question.

It has been uncertain whether for all d, there exist uniform antimatroid closure spaces $(\mathbf{U}, \varphi)$ of dimension d for $|\mathbf{U}| = n \geq d$. We spend the remainder of this paper demonstrating that this is so.

3 Uniform Closure Spaces of Higher Dimension

Figure 2 in the preceding section established that there exists a uniform closure space of dimension 2 over 5 elements. And, because we could observe that the closure space represents that obtained by interval closures over a totally ordered set, we could argue that such uniform closure spaces exist for all n. Figure 4 illustrates a uniform closure space of dimension 3 over 6 points. But, because we cannot identify an equivalent closure operator, we cannot assume that such closure spaces exist for all n.

We begin our argument by observing that every *generator-closure* pair is a boolean lattice of $2^{|Y.\varphi|-|Y.\beta|}$ elements. Hence, for all Y and all k such that $|Y.\beta| \leq k \leq |Y.\varphi|$, we

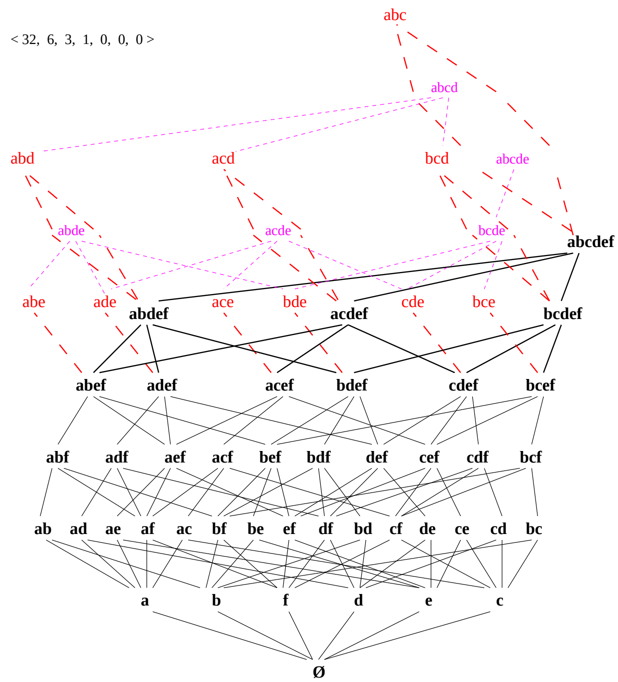

Figure 4: A uniform closure space, d=3

have

$$|\{Z \in [Y.\varphi, Y.\beta], |Z| = k\}| \quad = \quad C(|Y.\varphi| - |Y.\beta|, k - |Y.\beta|). \tag{3}$$

So the right hand combination counts the number of subsets Y in $[Y.\varphi, Y.\beta]$ with precisely k points. Further, if $|\mathbf{U}| = n$

$$\sum_{Y \ closed} 2^{|Y|-|Y.\beta|} = 2^n$$

because the lattice is an ordering of all the subsets of $\mathbf{U}$. If we let a_k denote the number of boolean intervals such that $|Y.\varphi| - |Y.\beta| = k$, (4) can be rewritten in the form

$$\sum_k a_k \cdot 2^k = 2^n. \tag{4}$$

A sequence of coefficients $< a_0, \ldots, a_k, \ldots, a_n >$ satisfying (4) are called the **partition coefficients** of the closure space. In [12, 13], an algorithm was presented to generate at least one corresponding closure lattice with that sequence. Hence (4) characterizes closure lattices in general, but not in particular. Distinct closure lattices can have identical coefficient sequences. Figure 1(b) has a coefficient sequence $< 22, 5, 3, 2, 0, 0, 0 >$. $< 10, 3, 2, 1, 0, 0, 0 >$ are the partition coefficients of Figure 2(b).

If we let b_k^d denote the number of non-simple closed sets with k elements, then for uniform closure spaces, (4) can be rewritten as

$$\sum_{k=0}^{d-1} C(n,k) + \sum_{k=d}^{n} b_k^d \cdot 2^{k-d} = 2^n. \tag{5}$$

If $(\mathbf{U}, \varphi)$ is uniform it is not hard to determine what these b_k^d must be.

Lemma 3.1 *Let $\mathcal{L}_{(U,\varphi)}$ be the lattice of an uniform closure space of dimension d on n points, then*

$$b_k^d = \begin{cases} 1 & k = n \\ C(n,k) - \sum_{i=k+1}^{n} C(i-d, k-d) b_i^d & d \le k < n \\ C(n,k) & k < d \end{cases}$$

Proof: The results for $k = n$ and $k < d$ are straight forward from the definition of uniformity and the structure of the lattice.

There are $C(n,k)$ sets of k elements. By definition, b_k^d of them are closed, and the rest are not. Consider all elements of height $i > k$. There are b_i^d elements, and by (3), for each such element Y, there are $C(i-d, k-d)$ elements of size k whose closure is Y. Thus there are $C(i-d, k-d)b_i^d$ elements of size k whose closure is of size i for $i > k$ (and clearly none for $i < k$). Thus $b_k^d + \sum_{i=k+1}^{d} C(i-d, k-d) b_i^d = C(n,k)$, and the result follows. □

If one evaluates the expression of this lemma, we obtain Table 1, where we count down from the largest closed set, $\mathbf{U}$. For many arguments, this *counting down* simplifies the understanding. For this reason, we say the **depth** of a closed element Y, denoted $dpt(Y)$, is $n - ht(Y)$.

Surprisingly, we observe that all uniform closure spaces of dimension d over n points, even non-isomorphic ones, must have identical numbers of closed sets at the same depth, and hence of the same size.

Corollary 3.2 *For any uniform closure space, the value of b_{n-k}^d is dependent only the values of k and d, hence independent of n and the structure of the closure space.*

Proof: Immediate from Lemma 3.1 □

The truth of this is illustrated in Figure 5 by a second, non-isomorphic uniform closure space of dimension d over 6 points. In both cases, $b_n^d = b_6^3 = 1$, $b_{n-1}^d = b_5^3 = 3$, $b_{n-2}^d = b_4^3 = 6$, and $b_{n-3}^d = b_3^3 = 10$ satisfying both Lemma 3.1 and Table 1. To assure ourselves that these

b^d_{n-k}	d 1	2	3	4	5	6	7
n	1	1	1	1	1	1	1
n-1	1	2	3	4	5	6	7
n-2	1	3	6	10	15	21	28
n-3	1	4	10	20	35	56	84
n-4	1	5	15	35	70	126	210
n-5	1	6	21	56	126	252	462
n-6	1	7	28	84	210	462	924
n-7	1	8	36	120	330	792	1,716
n-8	1	9	45	165	495	1,287	3,003

Table 1: b^d_{n-k}, $1 \leq d \leq 7$

two closure spaces, whose non-simple closed sets have identical structure, are really non-isomorphic, we observe that in Figure 5, $|\{def\}.\!\uparrow|$, the number of lattice elements covering $\{def\}$, is 3. It is the only such element of height 3. However there are 3 such elements in Figure 4, namely $\{aef\}, \{def\}$ and $\{cef\}$.

Thus, we have shown the existence of at least two uniform closure spaces of dimension 3, but we have yet to establish that they exist for all (d, n) pairs. For that we provide a construction.

Algorithm 3.3 *For any set of points,* $\mathbf{U} = \{p_1, p_2, ..., p_n\}$, *and any integer* $d, 2 \leq d < n$, *we designate the collection of closed sets of* $\mathbf{U}$ *by:*

(1) designate $\mathbf{U}$ as closed, with $\mathbf{U}.\beta = \{p_1, p_2, ..., p_d\}$;
Insert $\mathbf{U}$ into the collection of UNEXAMINED closed sets;

(2) for each set Y in UNEXAMINED do
- (a) remove Y from UNEXAMINED insert into CLOSED;
- (b) let $k = (dpt(Y) + 1) + d$;
- (c) for each $p_i \in Y.\beta$ do // p_k replaces each p_i
  - (1) let $X = Y - \{p_i\}$ be closed with $X.\beta = (Y.\beta - \{p_i\}) \cup \{p_k\}$;
  - (2) if $X \geq d$, insert X into UNEXAMINED

(3) For all $Y \in 2^{\mathbf{U}}, |Y| < d$ insert Y into CLOSED.

We have yet to show that this collection of designated closed sets and their generators constitute an antimatroid closure space.

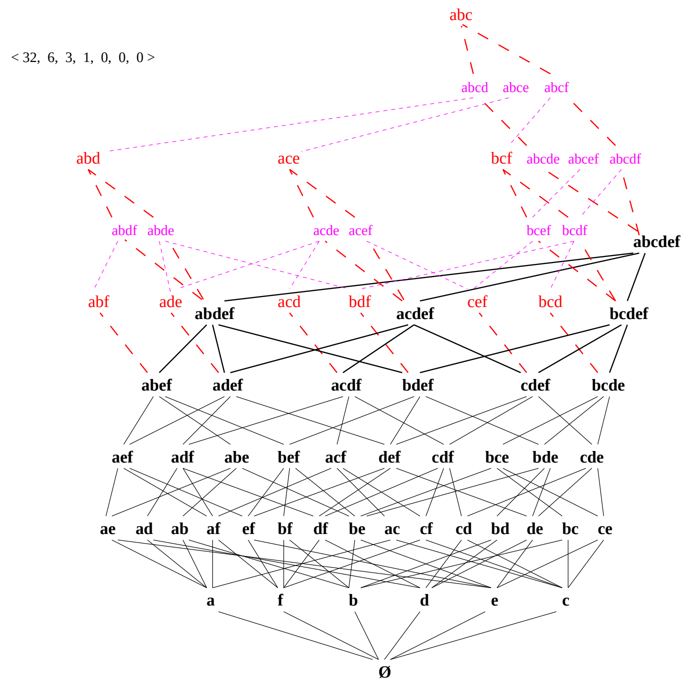

Figure 5: Another uniform closure space, d=3

Lemma 3.4 *For all closed sets Y generated by Algorithm 3.3 with $ht(Y) \geq d$, $Y.\beta = Y \cap \{p_1, p_2, ..., p_k\}$ where $k = n - ht(Y) + d$.*

Proof: Let $\mathbf{U}_k = \{p_1, p_2, ..., p_k\}$. It follows from the construction and simple induction that $Y.\beta \subseteq Y \cap \mathbf{U}_k$, where $k = dpt(Y) + d = n + d - ht(Y)$. So we need only show that $Y \cap \mathbf{U}_k \subseteq Y.\beta$. Readily this is true for the single closed set $\mathbf{U}$ at depth $= 0$. Let Y be any closed set at depth $= k = n - |Y|$. Choose any $y \in Y \cap \mathbf{U}_k$. If $y = p_k$, then by construction, $y \in Y.\beta$. If not, let Z be any closed set that gave rise to Y in the algorithm, that is $Y = Z - \{q\}, q \in Z.\beta$. $y \in Z$ and $y \neq p_k$ implies $y \in Z \cap U_{k-1}$ so by induction $y \in Z.\beta$. But, $y \in Y$ implies $y \neq q$, hence $y \in Z.\beta - \{q\}$ and by construction $y \in Y.\beta$. $\square$

Corollary 3.5 *For all closed sets Y generated by Algorithm 3.3, $Y.\beta$ is unique.*

Algorithm 3.3 only designated the closed sets of the space. We define the closure operator, φ, and its generator β by:

(1) For all $X \in 2^{\mathbf{U}}$, $X.\varphi = Y$, where $Y \in$ CLOSED and $Y.\beta \subseteq X \subseteq Y$.

(2) For all $X \in 2^{\mathbf{U}}$, $X.\beta = X.\varphi.\beta$.

Lemma 3.6 *φ, as defined above, is a well-defined operator.*

Proof: $Y.\varphi$ is defined for all Y.
$\mathbf{U} \in$ CLOSED and $Y \subseteq \mathbf{U}$. So, if $\mathbf{U}.\beta \subseteq Y$, $Y.\varphi = \mathbf{U}$.
Otherwise, let $p_1 \in \mathbf{U}.\beta - Y$ By construction, $Z_1 = \mathbf{U} - \{p_1\} \in$ CLOSED and $Y \subseteq Z_1$. If $Z_1.\beta \subseteq Y$, then $Y.\varphi = Z_1$ and we are done. Otherwise, we let $p_2 \in Z_1.\beta - Y$ and repeat this reduction step. If this continues until $|Z_i| < d$, then $|Y| < d$, implying $Y \in$ CLOSED and $Y.\beta = Y$.

$Y.\varphi$ is unique for all Y.
Assume $Y.\varphi = Z_1$ and $Y.\varphi = Z_2$, where both are incomparable. We may also assume that these are minimal such sets. Readily, $Y \subseteq Z_1 \cap Z_2$. If $(Z_1 \cap Z_2).\beta \subseteq Y$, then $Y.\varphi = Z_1 \cap Z_2$, contradicting the assumption of minimality. But, if $(Z_1 \cap Z_2).\beta \not\subseteq Y$, then there exists $p_i \in (Z_1 \cap Z_2).\beta - Y$ such that $Y \subseteq Z_1 - \{p_i\}$, leading to a similar contradiction of minimality.
Hence, Z_1 and Z_2 cannot be incomparable. Suppose $Z_1 \subset Z_2$. Consequently, Z_1 can be derived according to the algorithm from Z_2, in particular $\exists p_i \in Z_2.\beta - Z_1.\beta$. But, this immediately contradicts the relation $Z_2.\beta \subseteq Y \subseteq Z_1$. $\Box$

Theorem 3.7 *$(\mathbf{U}, \varphi)$, whose closed sets were designated by Algorithm 3.3, is a uniform closure space of dimension d.*

Proof: φ is a closure operator because, (a) $Y \subseteq Y.\varphi$ by construction, and (b) $X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$ because clearly $X \subseteq Y.\varphi$. If $Y.\beta \subseteq X$ then $X.\varphi = Y.\varphi$. Otherwise we can remove elements $p_i \in Y.\beta - X$ from Y as in the reduction of Lemma 3.4 to obtain $Y_i \subset Y$ such that $Y_i.\beta \subseteq X \subseteq Y_i$. Finally (c) $Y.\varphi.\varphi = Y.\varphi$ follows from the construction.

φ is uniquely generated. Suppose $X.\varphi = Y.\varphi$. Let $Z = X.\varphi$. By definition $Z.\beta \subseteq X, Y \subseteq Z$, hence $Z.\beta \subseteq X \cap Y \subseteq Z$ implying $(X \cap Y).\varphi = Z = X.\varphi$.

Finally, $(\mathbf{U}, \varphi)$ is uniform, because by construction $|Y.\beta| = d$ for all $Y, |Y| \geq d$ and Y is closed for all $Y, |Y| < d$. $\Box$

It can be valuable to follow this construction with an actual example. If we label the 6 points of a space by a, b, c, d, e and f, then the first step of the algorithm is illustrated by Figure 6. We have chosen a, b and c to be $\mathbf{U}.\beta$, hence $\mathbf{U}.\!\downarrow = \{abdef, acdef, bcdef\}$ with generators $ab\_$, $ac\_$ and $bc\_$ respectively. Notice that we have labeled each covering edge with the generator p_i whose removal gives rise to it. Lower semi-modularity, or the simple fact that the intersection of closed sets must be closed, forces the remaining closed sets of this illustration. The meet-distributive property of lower semi-modular lattices is amply illustrated as well.

The algorithm always chooses the point d to be p_4, the remaining generator. If we do so we obtain the partially completed lattice of Figure 7 for the next step. Here, each of the resulting closed sets that are forced by this decision are emboldened. It will be instructive

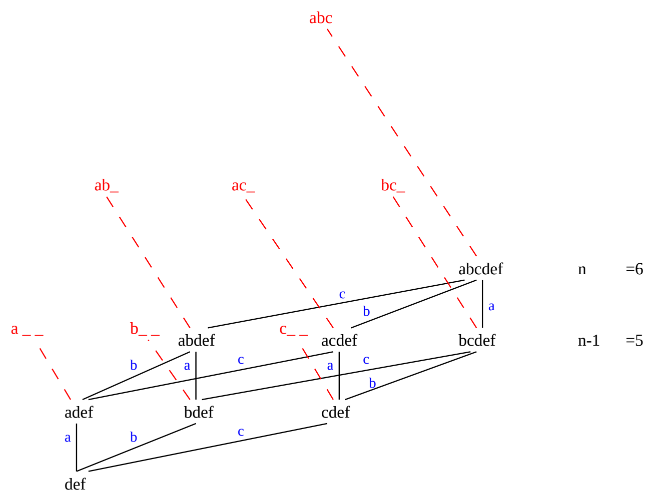

Figure 6: First step in the construction of a uniform closure space, d=3

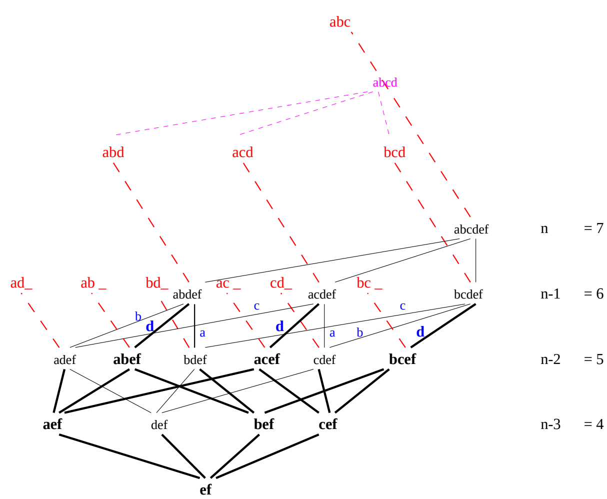

Figure 7: Second step in the construction of a uniform closure space, d=3

to complete the construction by assigning e to the generating sets of every element Y of height 5, and then making all sets of 2, or fewer elements, closed. The result should be the uniform space of Figure 4.

Observe that the algorithm always selects the same point p_k to complete the generating sets at each level. It simplifies the algorithm. But, it is not necessary, as Figure 5 attests. One can employ a non-deterministic generating algorithm.

We conclude this section with a lemma that characterizes the generating sets of uniform closure spaces generated by Algorithm 3.3. This, then provides a way of counting the numbers of closed sets at any height, or depth.

Lemma 3.8 *Let* $(\mathbf{U}, \varphi)$ *be a uniform closure space generated by 3.3. Then* X *is a generating set for some element* Y *where* $ht(Y) \geq d$ *if and only if*

(a) $p_k \in X$, *where* $k = dpt(Y) + d$,

(b)$|X.\beta| = d, and$

(c)$|X.\beta| \subseteq \{p_1, p_2, ..., p_k\}$

are true.

Proof: $(\Rightarrow)$ Follows directly from Theorem 3.7.
$(\Leftarrow)$ We run an induction on $dpt(X.\varphi)$. The result is clearly true when $dpt(X.\varphi) = 0$, because $X.\varphi = \mathbf{U}$, hence the only set that meets conditions (a), (b) and (c) is $\{p_1, p_2, ..., p_d\}$, the generator for $\mathbf{U}$.
Assume it is true for all X with $dpt(X.\varphi) < k$. Let Y be some set such that $dpt(Y.\varphi) = k$ and Y satisfies (a), (b) and (c). Since $|Y.\beta| = d$ and $dpt(Y.\varphi) > 0$, $\{p_1, p_2, ..., p_k\} - Y$ must be non-empty. If p_{k-1+d} is in the difference, let a be that point; else let a be any point in the difference. Let $Y' = (Y - \{p_{k-1}\}) \cup \{a\}$. By induction hypothesis, Y' is a generating set and $dpt(Y'.\varphi) = dpt(Y) - 1$. Now consider the set $(Y'.\varphi - \{a\}) \cup \{p_{k-1}\}$, which must be closed by the definition of the algorithm. Its generating set is $(Y' - \{a\}) \cup \{p_{k-1}\} = Y$, completing the induction. □

Theorem 3.9 *If* $(\mathbf{U}, \varphi)$ *is a uniform antimatroid closure space of dimension* d*, then for all* $0 \leq k \leq n - d$*,*

$$b^d_{n-k} = C(k + d - 1, d - 1)$$

Proof: By Corollary 3.2 we know the value of b^d_{n-k} is dependent only on d and k. So we need only compute the coefficients for the closure space generated by Algorithm 3.3, and this result will apply to all d-dimensional uniform closure spaces.
b^d_{n-k} is the number of closed sets Y such that $dpt(Y) = k$, which is equal to the number of distinct generating sets for such Y. Algorithm 3.3 always includes p_k, so it only chooses $d - 1$ points from the available generators used at depth $k - 1$. Consequently, by Lemma 3.8, the number of such generating sets X is the number of ways $d - 1$ points can be chosen from the set $\{p_1, p_2, ..., p_{k-1+d}\}$, which is $C(k + d - 1, d - 1)$. □

4 Decomposition of Uniform Closure Spaces

There is another way of deriving the b^d_{n-k}. Every uniform space of dimension d can be decomposed into uniform subspaces of dimension $d - 1$. In Figure 8, we illustrate the top

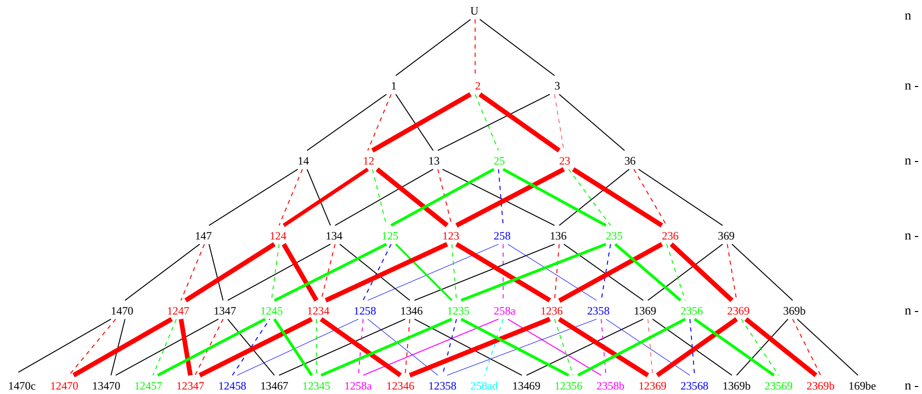

Figure 8: Six levels of a uniform closure space, d=3

six levels of a 3 dimensional, uniform closure space on n atoms. (The integers identifying the lattice elements are completely artificial, and have no significance.) The top three levels of this lattice are completely isomorphic to those of Figures 4 and 5.

This closure lattice can be decomposed into uniform closure spaces of dimension 2. The individual subspaces are drawn with solid lines. Their nesting is indicated by the dashed lines. We have tried to separate these nested 2-spaces by drawing them with solid lines of different thicknesses and intensity. Two of the elements covered by U, labeled 1 and 3, begin the first 2-dimensional subspace. The elements labeled 14, 13, 36 constitute the next level. Readily, this sublattice is isomorphic to the 2-dimensional, uniform lattice of Figure 2. The third element covered by U is 2. It will be the root of another 2-dimensional, uniform subspace with elements 12, 23, 124, 123, and 236. Each of these elements is covered by a corresponding element in the first sublattice, as indicated by the dashed lines. The third sublattice has 25 as its root. Each of its elements is covered by one in the second sublattice. The chain of roots of each of these nested sublattices is $U, 2, 25, 258, 258a, 258ad$.

This kind of decomposition provides another easy way of counting b^d_{n-k}, the number of closed subsets with $n-k$ elements (because $\mathcal{L}$ is atomic, every element of the $n-k$ level has $n-k$ atoms) in a d-dimensional space. We observe a general recurrence relation

$$b^d_{n-k} = b^d_{n-k+1} + b^{d-1}_{n-k}.$$

as well as

$$b^d_{n-k} = \sum_{i=n-k}^{n} b^{d-1}_i.$$

Using this recurrence, it is easy to reconstruct Table 1 for b^d_{n-k} for arbitrary d. And from this table, it is again evident that

$$b^d_{n-k} = C(k+d-1, d-1).$$

as asserted by Theorem 3.9.

5 Post Script

In [5], Erdos and Szekeres conjectured that a planar convex geometry of more than 2^{n-2} points must contain a convex n-gon. The n points may be a simple closed convex set, or may include other points in its interior, in which case it is a non-simple generating set. Readily, this conjecture is refuted if we can construct a closure space of dimension $n-1$ over $2^{n-2}+1$ atoms that is realizable as a planar convex geometry, and confirmed if no such realization is possible. Since the construction of the uniform closure space of dimension $n-1$ has been demonstrated, and since all $n-1$ dimensional spaces can be derived from the uniform $n-1$ dimensional space, the only issue is one of realizability.

As observed earlier, whether a closure space is realizable as a convex geometry, or a chordal graph, or a block graph, or some other configuration remains an open question. Nevertheless, we have an interesting relationship to a timeless conjecture.

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