Orders Induced by Closure Operators

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Abstract:

Any closure operator ϕ on a set S can be made to induce a partial order $\leqslant_\phi$ on its power set $P(S)$. In this paper we describe the properties of such an induced order. In particular, we will show that given minimal constraints on ϕ, $(P(S), \leqslant_\phi)$ is a semi-modular lattice. If S is itself a partially ordered set, it is shown that $\leqslant_\phi$ will "preserve" the order $\leqslant$ on S if and only if ϕ is the ideal operator on $(S, \leqslant)$. This development is a generalization of several other approaches to the structure of partially ordered sets and graphs, whose results fall out as corollaries.

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1. Background

Originally, the authors sought to answer the question "suppose $(S, \leqslant)$ is a partially ordered set, can the partial order relation be extended to $\boldsymbol{P}(S)$ in a 'natural' way?". This question appears in a number of applications such as [Day85] and [Esta80]. Similarly, in computer science one often seeks to partition a network into subsets that in some inherent fashion preserves the underlying structure of the network. The answer, which will appears as a proposition near the middle of this paper, is "yes". However, in the course of obtaining this result, a number of more fundamental relationships between closure operators (on either ordered or unordered sets) and partial orders on the power set were uncovered. These are covered in the next section.

The major result, showing that the partial order relation induced by a closure operator is a lower semi-modular lattice, is then developed in section 3. Demonstration of the same semi-modular lattice, but with respect to a particular "closure concept", are major steps in both [Koh84] and [Pfal73].

In a variety of computer science problems, it is often easier to define a closure concept than to demonstrate an explicit partial order. Instances include denotational semantics [Plot76], distributed concurrency control [Kane85], and communications protocols [Choi85]. That, under such situations, a partial order with regular lattice properties can be induced is often important. The authors, in particular, are using a closure operator and its induced semi-modular lattice to attack the classic grammar acquisition problem.

2. Partial Orders on a Power Set

Let S be any set with power set $P(S)$ consisting of all subsets of S. Let $\phi : P(S) \to P(S)$ be a closure operator on $P(S)$. Recall that a function ϕ is a **closure operator** if the following three closure axioms hold for all X, Y $\subseteq$ S

C1	$X \subseteq \phi(X)$
C2	$X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$
C3	$\phi(\phi(X)) = \phi^2(X) = \phi(X)$

In addition to these three standard closure properties, one can impose extra restrictions on the ϕ operator as follows

C4'	$\phi(X) = \phi(Y)$ implies $\phi(X \cap Y) = \phi(X) = \phi(Y)$
C5'	$\phi(X \cap Y) = \phi(X) \cap \phi(Y)$
C6'	$\phi(X \cup Y) = \phi(X) \cup \phi(Y)$

The first of these additional restrictions, which asserts that if two sets have the same closure then their intersection must as well, is relatively weak. It is frequently satisfied. However, it will be required later to insure that maximal elements exist in an induced lattice. The latter two conditions, that ϕ preserve unions and/or intersections, are rather strong restrictions. By way of example, the only closure operator that can simultaneously satisfy both C5' and C6' is the $\cup_A$ operator defined by $\cup_A(X) = X \cup A$ where A is any fixed subset. Most commonly $A = \emptyset$ so that $\cup_A$ is just the "identity" operator. Few familiar closure operators possess either of these properties. We will make little use of these two conditions, except to point out an occasional stronger result when one or the other holds. Readily C5' implies C4'. Finally, we point out that the two relationships

$$\phi(X) \cup \phi(Y) \subseteq \phi(X \cup Y)$$
$$\phi(X \cap Y) \subseteq \phi(X) \cap \phi(Y)$$

which follow directly from C2 are used repeatedly.

Let S be any set and let ϕ be any closure operator defined on S. For all X, Y $\subseteq$ S we say that $\mathbf{X} \leqslant_\phi \mathbf{Y}$, if

$$Y \cap \phi(X) \subseteq X \subseteq \phi(Y). \qquad (2.1)$$

We call this the **order induced on $P(S)$ by ϕ.** Of course, we must first show that $\leqslant_\phi$ so defined really is an order relation.

Theorem 2.1: $\leqslant_\phi$ is a partial order relation on $P(S)$.

Proof: We use the definition $X \leqslant_\phi Y \equiv_{def} Y \cap \phi(X) \subseteq X \subseteq \phi(Y)$ repeatedly in all of the proofs of this section. We provide a bit more detail for this first typical proof. For example, using C1 we have for all X, $X \cap \phi(X) \subseteq X \subseteq \phi(X)$ implying that $X \leqslant_\phi X$, or reflexivity.

If $X \leqslant_\phi Y$ and $Y \leqslant_\phi X$ then $Y \subseteq \phi(X)$ and $Y \cap \phi(X) \subseteq X$ which implies that $Y \subseteq X$. Similarly $X \subseteq Y$ so that $X = Y$ and weak anti-symmetry follows.

Let $X \leqslant_\phi Y$ and $Y \leqslant_\phi Z$. $X \leqslant_\phi Y$ implies $X \subseteq \phi(Y)$ so that by (C2) $Z \cap \phi(X) \subseteq Z \cap \phi(Y) \subseteq Y$ because $Y \leqslant_\phi Z$. Consequently, $Z \cap \phi(X) \subseteq Y \cap \phi(X) \subseteq X$. Further $X \subseteq \phi(Y)$ and $Y \subseteq \phi(Z)$ imply that $X \subseteq \phi\phi(Z) = \phi(Z)$. So $X \leqslant_\phi Z$ yielding transitivity. □

We are so accustomed to thinking of X as "smaller" (or less than) Y if $X \leqslant_\phi Y$, that one expects $X \subseteq Y$ to imply $X \leqslant_\phi Y$, or conversely. But this is not, in general, true. The following proposition develops relationships that do hold between the induced partial order, $\leqslant_\phi$, and set containment, $\subseteq$, in S.

Proposition 2.2: Let X, Y, Z $\subseteq$ S and let ϕ be a closure operator on S.
(1) $X \subseteq Y \subseteq Z$ and $X \leqslant_\phi Z$, imply $X \leqslant_\phi Y$
(2) $X \subseteq Y \subseteq Z$ and $Z \leqslant_\phi X$, imply $Z \leqslant_\phi Y$ and $Y \leqslant_\phi X$

Proof: Let $X \subseteq Y \subseteq Z$

(1) $X \subseteq Y$ implies $X \subseteq \phi(Y)$. Since $X \leqslant_\phi Z$, $Z \cap \phi(X) \subseteq X$, which with $Y \subseteq Z$ implies $Y \cap \phi(X) \subseteq X$, so $X \leqslant_\phi Y$. (Note that $Y \leqslant_\phi Z$, need not in general be true since $Z \cap \phi(Y)$ need not be contained in Y.)

(2) $X \subseteq Y$ implies $X \cap \phi(Y) \subseteq Y$, and $Y \subseteq Z$ implies $Y \cap \phi(Z) \subseteq Z$. From $Z \leqslant_\phi X$ we have $Z \subseteq \phi(X)$ which yields $Z \subseteq \phi(Y)$ from $X \subseteq Y$ and (C2); and $Y \subseteq \phi(X)$ from $Y \subseteq Z$. □

Notice that we can use containment properties to infer ordering relationships only in conjunction with other information concerning the induced order. Later theorem 2.5 will expand this theme to establish covering relationships based on single element membership and the closure operator, ϕ.

Unions and intersections of subsets of S can be related to $\leqslant_\phi$ by the following two propositions.

Proposition 2.3: Let $X, Y, Z \subseteq S$ and let ϕ be a closure operator on S.

(1) $X \leqslant_\phi Y$ and $X \leqslant_\phi Z$ imply $X \leqslant_\phi Y \cup Z$

(2) $X \leqslant_\phi Z$ and $Y \leqslant_\phi Z$ imply $X \cap Y \leqslant_\phi Z$

If ϕ preserves intersections, that is C5' above, then

(1') $X \leqslant_\phi Y$ and $X \leqslant_\phi Z$ imply $X \leqslant_\phi Y \cap Z$

while if ϕ preserves unions, that is C6' above, then

(2') $X \leqslant_\phi Z$ and $Y \leqslant_\phi Z$ imply $X \cup Y \leqslant_\phi Z$

Proof: (1) and (1').
$X \leqslant_\phi Y$ implies $X \subseteq \phi(Y)$, and similarly we have $X \subseteq \phi(Z)$. Thus $X \subseteq \phi(Y) \cup \phi(Z)$. (And $X \subseteq \phi(Y) \cap \phi(Z)$.) From the former, $X \subseteq \phi(Y \cup Z)$. Also $Y \cup \phi(X) \subseteq X$ and $Z \cup \phi(X) \subseteq X$, implying $(Y \cup Z) \cap \phi(X) \subseteq X$; hence $X \leqslant_\phi Y \cup Z$.

If ϕ preserves intersections then $X \subseteq \phi(Y \cap Z)$; moreover $Y \cap Z \subseteq Y \cup Z$ insures the second containment. Thus $X \leqslant_\phi Y \cap Z$.

(2) and (2') are demonstrated similarly. □

Proposition 2.4: Let $X, Y, Z \subseteq S$ and let ϕ be a closure operator on S.

(1) $X \leqslant_\phi Y$ implies $X \leqslant_\phi X \cup Y \leqslant_\phi Y$

(2) $X \leqslant_\phi Y$ implies $X \cap Y \leqslant_\phi Y$

(3) $X \leqslant_\phi Y \leqslant_\phi Z$ implies $X \cap Z \subseteq Y$

Proof: The first inequality in (1) and all of (2) follow directly from the preceding proposition, and the reflexivity of $\leqslant_\phi$. $X \cup Y \leqslant_\phi Y$ because $Y \cap \phi(X \cup Y) \subseteq X \cup Y$ trivially, and $X \cup Y \subseteq \phi(Y)$ because $Y \subseteq \phi(Y)$ and $X \leqslant_\phi Y$ implies that $X \subseteq \phi(Y)$. In (3), $X \leqslant_\phi Y$ implies $X \subseteq \phi(Y)$, consequently $X \cap Z \subseteq Z \cap \phi(Y) \subseteq Y$, (since $Y \leqslant_\phi Z$). □

Z is said to **cover** X if $X \leqslant_\phi Z$ and there exists no distinct Y such that $X \leqslant_\phi Y \leqslant_\phi Z$. The following theorem provides conditions under which a subset X of S can be covered by a subset Y that has exactly one more, or one less, element. It is fundamental to several dimension results, and also yields the immediate corollary that sets which differ by exactly one element are always comparable.

Theorem 2.5: If $x \notin X$ then

(1) $X \leqslant_\phi X \cup \{x\}$ if and only if $x \notin \phi(X)$

(2) $X \cup \{x\} \leqslant_\phi X$ if and only if $x \in \phi(X)$

where (1) is a cover if and only if $\phi(X \cup \{x\}) \subseteq \phi(X) \cup \{x\}$, and

(2) is always a covering relationship.

Moreover, if ϕ on S satisfies C4' then (1) and (2) characterize all covering relations in $(P(S), \leqslant_\phi)$.

Proof:

(1) Readily $X \subseteq \phi(X \cup \{x\})$; thus $X \leqslant_\phi X \cup \{x\}$ iff $(X \cup \{x\}) \cap \phi(X) \subseteq X$ iff $x \notin \phi(X)$. The issue is to establish the covering relationship. Let $\phi(X \cup \{x\}) \subseteq \phi(X) \cup \{x\}$ and let Y be such that $X \leqslant_\phi Y \leqslant_\phi X \cup \{x\}$. By the preceding proposition, $X \subseteq Y$. We assume that $X \neq Y$, else we are done. For $y \in Y - X$, $y \notin \phi(X)$ since $X \leqslant_\phi Y$.

$Y \subseteq \phi(X \cup \{x\}) \subseteq \phi(X) \cup \{x\}$ by assumption. Thus, if $y \in Y-X$, $y \in \{x\}$ that is $y = x$. Hence X is covered by $X \cup \{x\}$ in $\leqslant_\phi$.

(2) Readily $X \cap \phi(X \cup \{x\}) \subseteq X$; thus $X \cup \{x\} \subseteq \phi(X)$ iff $x \in \phi(X)$.

Let Y be such that $X \cup \{x\} \leqslant_\phi Y \leqslant_\phi X$. Again $X \subseteq Y$. Assume $X \neq Y$. Let $y \in Y-X$. $Y \leqslant_\phi X$ implies $Y \subseteq \phi(X)$ so in particular $y \in \phi(X)$. $X \cup \{x\} \leqslant_\phi Y$ implies $Y \cap \phi(X \cup \{x\}) \subseteq X \cup \{x\}$. So $y \in X \cup \{x\}$. Thus $y = x$.

Now assume that C4' is satisfied, and that Y covers X. By the preceding proposition, we know $X \leqslant_\phi X \cup Y \leqslant_\phi Y$, and thus either $X=X \cup Y$ or $Y=X \cup Y$ by the covering property. Simplifying, either $Y \subseteq X$ or $X \subseteq Y$.
In the first case, suppose $\exists x \in X-Y$. Let $Z = Y \cup \{x\}$ so that $Y \subset Z \subseteq X$. Since $X \leqslant_\phi Y$, by Proposition 2.2, $X \leqslant_\phi Z \leqslant_\phi Y$. Thus by the covering assumption, $Z = X = Y \cup \{x\}$.
Consequently, case (2) of the proposition holds immediately.
For the case $X \subseteq Y$, assume that $|Y - X| \geqslant 2$. Our goal is to show that Y can not cover X. Suppose for some $y_i, y_j \in Y-X$ that $\phi(X \cup \{y_i\} = \phi(X \cup \{y_j\}$. By C4', $\phi(X \cup \{y_i\}) = \phi(X \cup \{y_j\}) = \phi(X)$. Let $Z = X \cup \{y_i\}$. $X \subset X \cup \{y_i\} \subset Y$. By proposition 2.2, $X \leqslant_\phi X \cup \{y_i\}$ and $Y \cap \phi(X \cup \{y_i\}) = Y \cap \phi(x) \subseteq X \subseteq X \cup \{y_i\} \subseteq \phi(Y)$. So $X \cup \{y_i\} \leqslant_\phi Y$ contradicting the covering assumption.

On the other hand, if $\phi(X \cup \{y_i\} \neq \phi(X \cup \{y_j\})$ for all i, j, then by the pigeon hole principle, for at least one y_i, $\phi(X \cup \{y_i\}) = \phi(X) \cup \{y_i\}$. Now apply case (1) to

$$X \subset X \cup \{y_i\} \subset Y$$

to establish $\leqslant_\phi$ and contradict the initial covering assumption.

Hence, if Y covers X either $|Y - X| = 1$ or $|X - Y| = 1$. □

For most closure operators, $\phi(X \cup \{x\}) \subseteq \phi(X) \cup \{x\}$ will not generally be true. But for many individual elements $x \in S$ and sets $X \in \mathcal{P}(S)$ it will be true, thus leading to the covering relationships in $\leqslant_\phi$.

It is customary to say that a set X is **closed** if $X = \phi(X)$. Of the following corollaries, the first is immediate, and the second is a well known fact about closure operators.

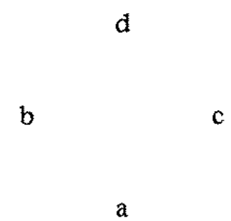
Corollary 2.6: If X is closed and $X \subseteq Y$ then $X \leqslant_\phi Y$.
Corollary 2.7: The arbitrary intersection of closed sets is closed.
Corollary 2.8: If Y is closed and X is covered by Y then $Y = X \cup \{x\}$ and X is closed. (The case $X = Y \cup \{x\}$ is impossible.)

Proof: Let Y cover X with respect to $\leqslant_\phi$, and let Y be closed.

If $Y = X \cup \{x\}$, then $Y = \phi(Y) = \phi(X \cup \{x\}) \subseteq \phi(X) \cup \{x\}$ by (1) of the preceding theorem. Hence $\phi(X) \subseteq \phi(X) \cap Y \subseteq X$ implying X is closed.

If $X = Y \cup \{x\}$ then by (2) $x \in \phi(Y)$. But this immeditely contradicts the assumption that Y is closed. □

There are many closure operators that can be defined on arbitrary sets (which need not be posets). To get some intutive sense of the nature of $\leqslant_\phi$ we consider a small partially ordered set S of 4 elements and three different closure operators ϕ. Let S be the poset



A partially ordered set $(S, \leqslant)$

First, observe that if ϕ is simply the **identity** operator, that is

$$\phi(X) = id(X) = X, \quad X \subseteq S$$

then ϕ satisfies all of the closure axioms including C4', C5', and C6'. In this case $(\boldsymbol{P}(S), \leqslant_\phi)$ is just the power set partially ordered by inclusion, $\subseteq$. It is worth reviewing all of the preceding results using this special case in which $\phi = id$ and $\leqslant_\phi = \subseteq$.

A second more interesting example is generated if we let ϕ be the familiar **ideal** operator, in which

$$\phi(X) = ideal(X) = \{ y : \exists x \in X, y \leqslant x \}$$

Again it is easy to verify the basic closure axioms. If S is finite, then readily C4' and C6' hold, but C5' does not. The induced partial order $(\boldsymbol{P}(S), \leqslant_\phi)$ is shown below. In the literature of denotational semantics this is called a Milner order [Plot76].

A third, and different, order $\leqslant_\phi$ is induced if we let ϕ be the **convex hull** operator, which can be denoted by **ch,** c.f. [Pfal71]. That is,

$$\phi(X) = ch(X) = \bigcap \{ Y : X \subseteq Y \text{ and } Y \text{ is convex in } S \}.$$

This too is a closure operator. It satisfies C4' and C5', but not C6'. It induces the d

ad bd cd

abd acd bcd

abcd bc

b abc c

ab ac

a

∅

The induced order $(\boldsymbol{P}(S), \leqslant_\phi)$
where $\phi(X)$ = ideal(X)

following partial order. In this paper, the semi-modular sub-lattice comprised of only the convex subsets of S was shown to have significent properties relative to the order, $\leqslant$, on S.

If S is itself a partially ordered set, as in the examples above, it is reasonable to want the induced partial order $\leqslant_\phi$ on $\boldsymbol{P}(S)$ to **preserve** the original partial order $\leqslant$ on S. That is, for all elements x, y $\in$ S

$$\{x\} \leqslant_\phi \{y\} \text{ if and only if } x \leqslant y.$$

Observe that for singleton elements of S, the relationship $\{x\} \leqslant_\phi \{y\}$ if and only if $x \in \phi(\{y\})$ is an immediate corollary of the definition of $\leqslant_\phi$. Consequently, if ϕ is an order perserving closure operator, then $x \leqslant y$ if and only if $x \in \phi(y)$. Equivalently, one may observe that ϕ is order preserving whenever the comparability graph of $(S, \leqslant)$ is an induced subgraph of the comparability graph of $(\boldsymbol{P}(S), \leqslant_\phi)$ in the sense of [Kell85] or [Gall68].

ad

abd acd

abcd

abc bcd

ab ac bc bd cd

a b c d

∅

The induced order $(\boldsymbol{P}(S), \leqslant_\phi)$
where $\phi(X) = ch(X)$

Proposition 2.9: If the induced partial order $\leqslant_\phi$ on $P(S)$ preserves the partial order $\leqslant$ on S then ϕ contains the ideal operator on S. If, in addition, ϕ preserves unions, C6′, then ϕ is the ideal operator.

Proof: Readily, if ϕ = ideal operator then ϕ preserves unions. Moreover, it is easily shown that $\leqslant_\phi$ on $\boldsymbol{P}(S)$ preserves $\leqslant$ on S in the sense that $x \leqslant y$ if and only if $\{x\} \leqslant_\phi \{y\}$.

Now let ϕ be any closure operator that preserves $\leqslant$ as above. In particular, we use the observation that $x \leqslant y$ iff $x \in \phi(\{y\})$. Let Y be any set, so ideal(Y) = $\{x \mid \exists y \in Y, x \leqslant y\} = \bigcup_{y \in Y} \phi(\{y\}) \subseteq \phi(\bigcup_{y \in Y} y) = \phi(Y)$ Thus, ideal(Y) $\subseteq \phi(Y)$.

And if ϕ satisfies C6′, the containment can not be proper. □

Given the ordering defined by (2.1), there is unique closure operator with will extend an ordering $\leqslant$ on S to an ordering $\leqslant_\phi$ on $\boldsymbol{P}(S)$. But there can be other extensions of $\leqslant$ to $\boldsymbol{P}(S)$. For example, a **dual** definition of the induced order using

$$X \leqslant_\phi Y \text{ if } X \cap \phi(Y) \subseteq Y \subseteq \phi(X).$$

can be of considerable practical interest. All of the preceding propositions (or slight variants) can be proven using this alternative dual definition. In this case, the unique closure operator that extends $\leqslant$ will be the **upper ideal** operator, which defines the set of all elements "greater than" a set X. Other "dual" properties are evident. "Duality" was first investigated in [Pfal73], which also first posed the question of extending partial orders from a set to its power set.

Often, one can define several arbitrary extensions to $\boldsymbol{P}(S)$ which preserve $\leqslant$, while satisfying the reflexive, anti-symmetric, and transitive properties expected of a partial order. A natural constraint on any extension of $\leqslant$ to $\boldsymbol{P}(S)$ would be to require that "if for all $x \in X$, there exists a $y \in Y$ such that $x \leqslant y$, then it would follow that $X \leqslant_{\phi} Y$". Based on the work in [Pfal73] it is conjectured that any extension of $\leqslant$ to $\boldsymbol{P}(S)$ which has this property must contain either $\leqslant_{\phi}$ or its dual as a sub order.

3. The Induced Lattice of Subsets

Before establishing the lattice properties of $(S, \leqslant_\phi)$, we first develop a concept of "maximal" elements with respect to the closure operator, ϕ. In fact, these are analogous to maximal elements with respect to the order, $\leqslant_\phi$, and they will be used in this fashion. This development, however, is a little bit cleaner and more basic.

Let X be any subset of S. The set of **maximal elements** with respect to X and ϕ on S is

$$\max_\phi(X) = \bigcap_k \{ Z_k : \phi(Z_k) = \phi(X) \}$$

One may also regard $\max_\phi(X)$ as the minimal set of generators for $\phi(X)$.

It may be easier to visualize $\max_\phi$ by considering the case where S is a poset and ϕ is the ideal operator. Then $\max_\phi(X)$ is precisely those elements which are maximal in X. (Had we developed the dual formulation, we would have called it $\min_\phi$.) It is clear that we have chosen our notation to continue this intuitive imagery, even when S is not partially ordered or ϕ is some other closure operator. While this intuitive view can be helpful, the $\max_\phi$ operator is really characterized by the following propositions.

Proposition 3.1: Let X, Y $\subseteq$ S and let ϕ be a closure operator on S.

(1) $\max_\phi(X) \subseteq X$

(2) $\phi(\max_\phi(X)) \subseteq \phi(X)$
with equality if ϕ satisfies C4'

(3) $\max_\phi(X) = \max_\phi(\phi(X))$

(4) $\max_\phi(X) = \max_\phi(\max_\phi(X))$ **if ϕ satisfies C4'.**

Proof: Part (1) is evident from the definition. Application of (C2) to (1) immediately yields (2). However, an alternative derivation may be more illustrative. $\phi(\max_\phi(X)) = \phi(\cap_k Z_k) \subseteq \cap_k \phi(Z_k) = \phi(X)$ where $\phi(Z_k) = \phi(X)$. Since for all Z_k, $\phi(Z_k) = \phi(X)$, with C4' equality readily follows.

$\max_\phi(\phi(X)) = \cap_k \{ Z_k : \phi(Z_k) = \phi(\phi(X)) = \phi(X) \}$ where the latter expression is just $\max_\phi(X)$.

$\max_\phi(\max_\phi(X)) = \cap_k \{ Z_k : \phi(Z_k) = \phi(\max_\phi(X))$ where using (2) yields $\max_\phi(X)$ for the latter term. □

Note that the condition C4', $\phi(X) = \phi(Y)$ implies $\phi(X \cap Y) = \phi(X) = \phi(Y)$, is very much less stringent that C5', the preservation of intersections by ϕ, which implies it. Relatively few closure operators satisfy C5', while the majority will satisfy C4'.

Proposition 3.2: Let X, Y $\subseteq$ S and let ϕ be a closure operator on S.

(1) $Y \subseteq X$ **implies** $Y \cap \max_\phi(X) \subseteq \max_\phi(Y)$

(2) $\max_\phi(X) \cap \max_\phi(Y) \subseteq X \cap \max_\phi(Y) \subseteq \max_\phi(X \cap Y)$

(3) $\max_\phi(X) \subseteq \{x \in X : x \notin \phi(X - \{x\})\}$
with equality if ϕ satisfies C4'.

Proof:

(1) Let W_j be the generators of $\max_\phi(Y)$, so that $\phi(W_j) = \phi(Y)$. It suffices to show that for all j, $Y \cap \max_\phi(X) \subseteq W_j$, or equivalently, that $\max_\phi(X) \subseteq W_j \cup (X-Y)$. In fact, we claim that $\phi(X) = \phi(W_j \cup (X-Y))$. To see this, note that $X = \phi(W_j) \cup (X-Y)$ so $X \subseteq \phi(W_j \cup (X-Y))$, thus $\phi(X) \subseteq \phi(W_j \cup (X-Y))$. Similarly, $W_j \cup (X-Y) \subseteq \phi(X)$, and thus $\phi(W_j \cup (X-Y)) \subseteq \phi(X)$, and we are done.

(2) The first inequality follows from $\max_\phi(X) \subseteq X$. By part (1), we have $X \cap Y \cap \max_\phi(Y) \subseteq \max_\phi(X \cap Y)$. But $X \cap Y \cap \max_\phi(Y) = X \cap \max_\phi(Y)$, so the second inequality is proven.

(3) Let $x \in \max_\phi(X)$. For any Z_k such that $\phi(Z_k) = \phi(X)$ we have $x \in Z_k$ implying $\phi(X - \{x\})$ is a proper subset of $\phi(X)$. So $x \notin \phi(X - \{x\})$ or $\max_\phi(X) \subseteq \{x \in X : x \notin \phi(X - \{x\})\}$.

Let us denote the latter set above by B, and let $A = \{x \in X : x \notin \phi(\phi(X) - \{x\})\}$. Now let $x \in A$. $x \in X$ and $x \notin \phi(\phi(X) - \{x\})$. Again let Z_k be such that $\phi(Z_k) = \phi(X)$. $Z_k \subseteq \phi(X)$. If $x \notin Z_k$ then $\phi(Z_k) \subseteq \phi(\phi(X) - \{x\})$, so $x \notin \phi(Z_k)$, a contradiction. So $x \in Z_k$ for all Z_k implying $x \in \max_\phi(X)$.

Thus we have proven that $A \subseteq \max_\phi(X) \subseteq B$, which is a slightly stronger result than asserted in the proposition itself.

Now, if (C4') holds we can show that A = B. Suppose $x \notin A$, we will show that $x \notin B$. $x \notin A$ implies either $x \notin X$, in which case $x \notin B$ follows immediately, or $x \in X$ and $x \in \phi(\phi(X) - \{x\})$. These imply $\phi(X) = \phi(\phi(X) - \{x\})$. By (C4'), $\phi(X) = \phi(X \cap (\phi(X) - \{x\})) = \phi(X - \{x\})$, thus $x \in \phi(X - \{x\})$ or $x \notin B$. □

Theorem 3.3: If ϕ satisfies C4', then $(P(S), \leqslant_\phi)$ is a lattice with

$$\inf(X_1, \ldots, X_n) = [(\bigcup_i X_i) \cap (\bigcap_i \phi(X_i))] \cup \max_\phi(\bigcap_i \phi(X_i))$$

Proof: Let $I = [(\cup_i X_i) \cap (\cap_i \phi(X_i))] \cup \max_\phi(\cap_i \phi(X_i))$. Since the finite intersection of closed sets, $\cap_i \phi(X_i)$, is closed, $\phi(I) = \cap_i \phi(X_i)$, because $\phi[(\cup_i X_i) \cap (\cap_i \phi(X_i))] \subseteq \cap_i \phi(X_i)$, and $\phi(\max_\phi(\cap_i \phi(X_i))] = \cap_i \phi(X_i)$.

Now, $I \subseteq \cap_k \phi(X_k)$ implies $I \subseteq \phi(X_k)$, while $X_k \cap \phi(I) \subseteq I$ follows from $X_k \cap \phi(I) = X_k \cap (\cap_i \phi(X_i)) \subseteq (\cup_i X_i) \cap (\cap_i \phi(X_i)) \subseteq I$. Thus, for all X_k, $I \leqslant_\phi X_k$.

Suppose now that for all k $Y \leqslant_\phi X_k$. Then $X_k \cap \phi(Y) \subseteq Y \subseteq \phi(X_k)$, so that $Y \subseteq \cap_i \phi(X_i)$, and $(\cup_i X_i) \cap \phi(Y) \subseteq Y$. We must show that $Y \leqslant_\phi I$, that is $\phi(Y) \cap I \subseteq Y \subseteq \phi(I)$. The latter containment was just shown, as well as $\phi(Y) \cap [\cup_i X_i \cap (\cap_i \phi(X_i))] \subseteq Y$.
It remains only to show that $\phi(Y) \cap \max_\phi(\cap_i \phi(X_i)) \subseteq Y$. By proposition 3.2, $\phi(Y) \cap \max_\phi(\cap_i \phi(X_i) \subseteq \max_\phi(\phi(Y) \cap (\cap_i \phi(X_i)))$. Now $\phi(Y) \subseteq \cap_i \phi(X_i)$, so

$\max_\phi(\phi(Y) \cap (\cap_i \phi(X_i))) = \max_\phi(\phi(Y)) = \max_\phi(Y) \subseteq Y.$

Having demonstrated that the finite inf operator exists we need only establish the existence of a maximal element. We claim it is $\max_\phi(S)$, where S is the entire set. Let $X \subseteq S$. $\phi(X) \cap \max_\phi(S) \subseteq \max_\phi(\phi(X) \cap S) = \max_\phi(\phi(X)) = \max_\phi(X) \subseteq X$ and readily $X \subseteq \phi(\max_\phi(S)) = S$. So $X \leqslant_\phi \max_\phi(S)$. □

Note that from the definition, $X \cap Y \subseteq \inf(X, Y)$.

The preceding theorem is conditioned on ϕ satisfying C4' on S. For the remainder of the paper we assume this is the case. If not, S may be partitioned into subsets $A_1, \ldots, A_n$ for which C4' is satisfied. The equivalence classes, A_k, of the finest such partition are called **atoms** of S with respect to the closure operator. (c.f. [Pfal71])

Theorem 3.4: The lattice $(P(S), \leqslant_\phi)$ is lower semi-modular.

Proof: Let Z cover X and Y in $(P(S), \leqslant_\phi)$, where X and Y are not comparable wrt. $\leqslant_\phi$.
Let $I = \inf(X, Y)$.
Case (1) $Z = Y \cup \{x\}$.
We will show that $X = I \cup \{x\}$. By theorem 2.5, $\phi(Y \cup \{x\}) \subseteq \phi(Y) \cup \{x\}$ and $x \notin \phi(Y)$ implying that $x \notin I$.

$I \subseteq X$. If $x \notin X$ then since $x \in Z$ and Z cover X, $Z = X \cup \{x\}$ or $X = Y$, a contradiction. So $x \in X$ and $I \cup \{x\} \subseteq X$.

If $X \supsetneq I \cup \{x\}$, there exists $y \in Y$, $y \notin I$. Either $y \in Z$ or it isn't. In the former case we derive a contradiction since $y \in Z$ implies $y \in Y$ implies $y \in I$. In the latter case, if $y \notin Z$ then $X = Z \cup \{y\}$ (since Z covers X) implying that $Y \subseteq Z \subseteq X \subseteq I$. $I \leqslant_\phi Y$ so by proposition 2.2, $X \leqslant_\phi Y$ contradicting non comparability.

Thus $X = I \cup \{x\}$, so X covers I.
Case (2) $Z = Y - \{x\}$.
One shows in a very similar manner that $X = I - \{x\}$. □

Proposition 3.5: Let $C \subseteq P(S)$ be the collection of all subsets of S which are closed with respect to ϕ. If ϕ satisfies C4' then $(P(S), \leqslant_\phi)$ restricted to C is a lower semi-modular sublattice $(C, \leqslant_\phi)$. Moreover, this sublattice of closed sets is always partially ordered by set containment, $\subseteq$.

Proof: The entire set S is closed. Since ϕ satisfies C4', all covering relationships are governed by theorem 2.5 and its corollaries.

Readily, if S covers X, then X is closed. By a simple induction one obtains that $X \leqslant_\phi S$ implies X is closed. Thus $(C, \leqslant_\phi)$ is the sublattice of all sets X, $X \leqslant_\phi S$ and it is readily semi-modular.

Moreover, only rule (1) of the theorem is ever applicable, so the partial order $\leqslant_\phi$ is equivalent to simple containment, $\subseteq$.

Alternatively, one can see that for closed sets X and Y, $\inf(X, Y) = X \cap Y$. □

Both [Koh84] and [Pfal71] describe the properties of certain subsets of a graph in terms of a lattice of such subsets. In both papers, proof of lower semi-modularity is a major step. Since Koh's "closed set" readily defines a closure operator, as does the the "convex hull" operator, these results could equally well be asserted as corollaries of the preceding theorem.

It is frequently fairly easy to demonstrate that closure properties C1 through C3 hold for interesting substructures of various discrete structures, such as directed and undirected graphs, or computer program units. With the preceding results, one can immediately induce a partial order on these sub-structures which is quite rich mathematically. There are a number of applications in which this partial order is of considerable interest.

We close the paper with a corollary that illustrates one such simple application. By definition, $\varnothing$ is always the least element of $(\mathcal{P}(S), \leqslant_\phi)$. Let the **height** of X in S, denoted ht(X), be the length of a chain from $\varnothing$ to X in $(\mathcal{P}(S), \leqslant_\phi)$. Because this lattice is semi-modular, it satisfies the Dedekind Chain Condition and all such chains have the same length. Thus height is a well defined concept.

Corollary 3.6: If ϕ satisfies C4' then in $(\mathcal{P}(S), \leqslant_\phi)$,

$$\mathbf{ht(X) = 2 \cdot |\phi(X)| - |X|.}$$

Proof: We must show that Y covers X iff $X \leqslant_\phi Y$ and $ht(Y) = ht(X) + 1$. This follows as a corrolary of theorem 2.5.

In the forward direction, theorem 2.5 gives us two cases. In case (1), $Y = X \cup \{x\}$, with $x \notin \phi(X)$ and $\phi(Y) = \phi(X \cup \{x\}) = \phi(X) \cup \{x\}$. Thus $|Y| = |X| + 1$, and $|\phi(Y)| = |\phi(X)| + 1$, so $ht(Y) = ht(X) + 1$.

In case (2), $X = Y \cup \{y\}$, with $y \in \phi(Y)$. Thus $\phi(X) = \phi(Y)$, and so $ht(X) = ht(Y) - 1$, and we are done.

For necessity, notice that the cardinality conditions above yield the containment conditions of theorem 2.5. □

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