Partitions of 2^n *

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Abstract

There are many ways that 2^n can be expressed as the sum of lower powers of 2, that is $\sum_{k=0}^{n} a_k \cdot 2^k = 2^n$, where a_k is a non-negative integer. Each collection of coefficients $< \cdots a_k \cdots >$ is a partition of 2^n. This paper presents a way of counting the number, p_n, of such partitions, which is super exponential.

1 Generating Partitions

By a **partition** of 2^n we mean a sequence of non-negative integers $< \cdots, a_k \cdots >$, $0 \leq k \leq n$ such that

$$a_0 \cdot 2^0 + a_1 \cdot 2^1 + a_2 \cdot 2^2 + \cdots + a_{n-1} \cdot 2^{n-1} + a_n \cdot 2^n = 2^n \tag{1}$$

or $\sum_{k=0}^{n} a_k \cdot 2^k = 2^n$. The set of all such partitions we denote by $\mathbf{P}^n$. Such partitions arise in the study of closure spaces, where it can be shown that every closure operator φ has a *trace* satisfying (1), and that for any given partition of 2^n, there exists a closure operator having that sequence as its trace [3].

Several characteristics of (1) are readily apparent. First, $a_n \neq 0$ if and only if $a_k = 0$ for all $0 < k < n$. Second, since the right hand side is even and all terms $a_k \cdot 2^k$, $k > 0$ must be even, the coefficient a_0 must be even. Third, if $< \cdots, a_{k-1}, a_k, \cdots >$ is a partition, then $< \cdots, a_{k-1} + 2, a_k - 1, \cdots >$ must be as well. And fourth, if $< a_0, \cdots, a_k, \cdots, a_n >$ is a partition of 2^n then $< 0, a_0, \cdots, a_k, \cdots, a_n >$ is a partition of 2^{n+1}.

With these observations, it is not difficult to write a process which generates all partitions in lexicographic order. (For this we represent the partition coefficients in reverse order $a_n, a_{n-1}, \cdots, a_1, a_0$.) The following C process, given a partition of 2^n returns the next one in the lexicographic order.

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```
int     next_n_part (int n, int *a[ ])
        /*
        ** Given a power set partition a[0],...,a[n], generate the
        ** next one in lexicographic order, and return 1.
        ** If no more exist, return 0.
        */
        {
        int     i, k;

        i = 1;                          // find lowest order, non-zero
        while (i < n && *a[i] == 0)     // term a[i], i > 0
                ++i;

        if (i == n && *a[n] == 0)
                return 0;               // last partition < 2**n, 0 ,..., 0 >

        if (i != 1 && *a[0] != 0)
                {
                *a[i]  -= 1;
                *a[i-1] = 2 + *a[0]/(2**(i-1));
                *a[0] = 0;
                }
        else
                {
                *a[i]   -= 1;
                *a[i-1] += 2;
                }

        return 1;
        }
```

Executing this procedure, and displaying each partition, generates the following enumerations of $\mathbf{P}^3$ and $\mathbf{P}^4$ (where a_n is displayed as the leading coefficient).

n = 3

1	0	0	0
0	2	0	0
0	1	2	0
0	1	1	2
0	1	0	4
0	0	4	0
0	0	3	2
0	0	2	4
0	0	1	6
0	0	0	8

n = 4

1	0	0	0	0
0	2	0	0	0
0	1	2	0	0
0	1	1	2	0
0	1	1	1	2
0	1	1	0	4
0	1	0	4	0
0	1	0	3	2
0	1	0	2	4
0	1	0	1	6
0	1	0	0	8
0	0	4	0	0
0	0	3	2	0
0	0	3	1	2
0	0	3	0	4
0	0	2	4	0
0	0	2	3	2
0	0	2	2	4
0	0	2	1	6
0	0	2	0	8
0	0	1	6	0
0	0	1	5	2
0	0	1	4	4
0	0	1	3	6
0	0	1	2	8
0	0	1	1	10
0	0	1	0	12

0	0	0	8	0
0	0	0	7	2
0	0	0	6	4
0	0	0	5	6
0	0	0	4	8
0	0	0	3	10
0	0	0	2	12
0	0	0	1	14
0	0	0	0	16

It is quite easy to verify by inspection that each sequence is a partition of 2^n. And because they are in lexicographic order, one can verify that all possible partitions have been generated.

If we let p_n denote $|\mathbf{P}^n|$, that is the number of distinct partitions of 2^n, then one can also verify that $p_3 = 10$ and $p_4 = 36$. For convenience in the remaining paper, we shall designate those partitions with $a_0 \neq 0$ as **normal partitions**.[1] Now, let y_n denote the number of *normal* partitions, and z_n denote the number of *non-normal* partitions in $\mathbf{P}^n$. Then by inspection, $y_3 = 6, y_4 = 26$ and $z_3 = 4, z_4 = 10$. And, readily, $p_n = y_n + z_n$.

There is a pattern developing in the sequence of a_0 coefficients which will turn out to be crucial for counting these partitions. Following a non-normal partition in which $a_0 = 0$ there will be a sequence of (possibly zero, whenever $a_1 = 0$) normal partitions whose a_0 coefficients are strictly increasing. We see this pattern, which we will exploit in the following section, emerging more clearly when we run the same program with $n = 5$.

n = 5	1	0	0	0	0	0	0	0	1	1	6	8
	0	2	0	0	0	0	0	0	1	1	5	10
	0	1	2	0	0	0	0	0	1	1	4	12
	0	1	1	2	0	0	0	0	1	1	3	14
	0	1	1	1	2	0	0	0	1	1	2	16
	0	1	1	1	1	2	0	0	1	1	1	18
	0	1	1	1	0	4	0	0	1	1	0	20
	0	1	1	0	4	0	0	0	1	0	12	0
	0	1	1	0	3	2	0	0	1	0	11	2
	0	1	1	0	2	4	0	0	1	0	10	4
	0	1	1	0	1	6	0	0	1	0	9	6
	0	1	1	0	0	8	0	0	1	0	8	8
	0	1	0	4	0	0	0	0	1	0	7	10
	0	1	0	3	2	0	0	0	1	0	6	12
	0	1	0	3	1	2	0	0	1	0	5	14
	0	1	0	3	0	4	0	0	1	0	4	16
	0	1	0	2	4	0	0	0	1	0	3	18
	0	1	0	2	3	2	0	0	1	0	2	20
	0	1	0	2	2	4	0	0	1	0	1	22
	0	1	0	2	1	6	0	0	1	0	0	24
	0	1	0	2	0	8	0	0	0	8	0	0
	0	1	0	1	6	0	0	0	0	7	2	0
	0	1	0	1	5	2	0	0	0	7	1	2
	0	1	0	1	4	4	0	0	0	7	0	4
	0	1	0	1	3	6	0	0	0	6	4	0

[1] There is some slight justification for regarding a partition of 2^n with $a_0 = 0$ as *non-normal*. While there exists at least one closure space on n points corresponding to every partition $< a_0, \cdots, a_n >$, it has been suggested that for the closure space to be a convex geometry [1] or an alignment [2] the empty set should be closed, or equivalently $a_0 \neq 0$. These, therefore, correspond to normal partitions. And closure spaces in which $\emptyset$ is not closed are non-normal.

0	1	0	1	2	8
0	1	0	1	1	10
0	1	0	1	0	12
0	1	0	0	8	0
0	1	0	0	7	2
0	1	0	0	6	4
0	1	0	0	5	6
0	1	0	0	4	8
0	1	0	0	3	10
0	1	0	0	2	12
0	1	0	0	1	14
0	1	0	0	0	16
0	0	4	0	0	0
0	0	3	2	0	0
0	0	3	1	2	0
0	0	3	1	1	2
0	0	3	1	0	4
0	0	3	0	4	0
0	0	3	0	3	2
0	0	3	0	2	4
0	0	3	0	1	6
0	0	3	0	0	8
0	0	2	4	0	0
0	0	2	3	2	0
0	0	2	3	1	2
0	0	2	3	0	4
0	0	2	2	4	0
0	0	2	2	3	2
0	0	2	2	2	4
0	0	2	2	1	6
0	0	2	2	0	8
0	0	2	1	6	0
0	0	2	1	5	2
0	0	2	1	4	4
0	0	2	1	3	6
0	0	2	1	2	8
0	0	2	1	1	10
0	0	2	1	0	12
0	0	2	0	8	0
0	0	2	0	7	2
0	0	2	0	6	4
0	0	2	0	5	6
0	0	2	0	4	8
0	0	2	0	3	10
0	0	2	0	2	12
0	0	2	0	1	14
0	0	2	0	0	16
0	0	1	6	0	0
0	0	1	5	2	0
0	0	1	5	1	2
0	0	1	5	0	4
0	0	1	4	4	0
0	0	1	4	3	2
0	0	1	4	2	4
0	0	1	4	1	6
0	0	1	4	0	8
0	0	1	3	6	0
0	0	1	3	5	2
0	0	1	3	4	4
0	0	1	3	3	6
0	0	1	3	2	8
0	0	1	3	1	10
0	0	1	3	0	12
0	0	1	2	8	0
0	0	1	2	7	2
0	0	1	2	6	4
0	0	1	2	5	6
0	0	1	2	4	8

0	0	0	6	3	2
0	0	0	6	2	4
0	0	0	6	1	6
0	0	0	6	0	8
0	0	0	5	6	0
0	0	0	5	5	2
0	0	0	5	4	4
0	0	0	5	3	6
0	0	0	5	2	8
0	0	0	5	1	10
0	0	0	5	0	12
0	0	0	4	8	0
0	0	0	4	7	2
0	0	0	4	6	4
0	0	0	4	5	6
0	0	0	4	4	8
0	0	0	4	3	10
0	0	0	4	2	12
0	0	0	4	1	14
0	0	0	4	0	16
0	0	0	3	10	0
0	0	0	3	9	2
0	0	0	3	8	4
0	0	0	3	7	6
0	0	0	3	6	8
0	0	0	3	5	10
0	0	0	3	4	12
0	0	0	3	3	14
0	0	0	3	2	16
0	0	0	3	1	18
0	0	0	3	0	20
0	0	0	2	12	0
0	0	0	2	11	2
0	0	0	2	10	4
0	0	0	2	9	6
0	0	0	2	8	8
0	0	0	2	7	10
0	0	0	2	6	12
0	0	0	2	5	14
0	0	0	2	4	16
0	0	0	2	3	18
0	0	0	2	2	20
0	0	0	2	1	22
0	0	0	2	0	24
0	0	0	1	14	0
0	0	0	1	13	2
0	0	0	1	12	4
0	0	0	1	11	6
0	0	0	1	10	8
0	0	0	1	9	10
0	0	0	1	8	12
0	0	0	1	7	14
0	0	0	1	6	16
0	0	0	1	5	18
0	0	0	1	4	20
0	0	0	1	3	22
0	0	0	1	2	24
0	0	0	1	1	26
0	0	0	1	0	28
0	0	0	0	16	0
0	0	0	0	15	2
0	0	0	0	14	4
0	0	0	0	13	6
0	0	0	0	12	8
0	0	0	0	11	10
0	0	0	0	10	12
0	0	0	0	9	14
0	0	0	0	8	16

0	0	1	2	3	10	0	0	0	0	7	18
0	0	1	2	2	12	0	0	0	0	6	20
0	0	1	2	1	14	0	0	0	0	5	22
0	0	1	2	0	16	0	0	0	0	4	24
0	0	1	1	10	0	0	0	0	0	3	26
0	0	1	1	9	2	0	0	0	0	2	28
0	0	1	1	8	4	0	0	0	0	1	30
0	0	1	1	7	6	0	0	0	0	0	32

When $n = 5, p_5 = 202, y_5 = 166$, and $z_5 = 36$.

Even for small n, $\mathbf{P}^n$ can be very large, as shown by Table 1 in which $\mathbf{P}^n$ was enumerated and the number of generated partitions counted.

n	$\lvert\mathbf{P}^n\rvert$
3	10
4	36
5	202
6	1,828
7	27,338
8	692,004

Table 1: Number, $|\mathbf{P}^n|$, of enumerated partitions of 2^n

2 Counting Partitions

One could continue generating all partitions and counting them using the techniques of the preceding section. But as is evident from in Table 1, this soon becomes computationally prohibitive. Instead, one seeks a simple recurrence relation that describes p_n in terms of $p_{n-1}, p_{n-2}, \cdots$. A recurrence exists, but it is far from simple.

As observed in the preceding section, a_0 must be even and if $< a_0, \cdots, a_{n-1} >$ is a partition in $\mathbf{P}^{n-1}$ then $< 0, a_0, \cdots, a_{n-1} >$ is a partition in $\mathbf{P}^n$. Consequently,

$$z_n = p_{n-1} \tag{2}$$

Since, $p_n = y_n + z_n$, our only problem is to recursively determine y_n, the number of normal partitions.

In the lexicographic order of $\mathbf{P}^n$, if $\pi_i^n =< 0, a_1, a_2, \cdots, a_n >\in \mathbf{P}^n, a_1 \neq 0$, then there must follow the sequence $S_{a_1}^n$ of partitions, $< 2, a_1 - 1, a_2, \cdots, a_n >$, $< 4, a_1 - 2, a_2, \cdots, a_n >$, $\cdots$, $< 2a_1, 0, a_2, \cdots, a_n >$. Readily, the length of this sequence $|S_{a_1}^n|$ is a_1. Hence, each normal partition $\pi_i^{n-1} \in \mathbf{P}^{n-1}$ gives rise to a subsequence of $a_1^n = a_0^{n-1}$ normal partitions in $\mathbf{P}^n$.

If one carefully keeps track of all normal permutations in $\mathbf{P}^{n-1}$, then one can use the mechanism above to generate all normal partitions in $\mathbf{P}^n$. This is illustrated in Figure 1 in which subsequences S_k^n of normal partitions are enumerated (by showing only the a_0

```
n =   1      2      3       4       5

                            2   ==>2 4
                    2  ==>  4   ==>2 4 6 8
                            2   ==>2 4
                            4   ==>2 4 6 8
                            6   ==>2 4 6 8 10 12
             2  ==>4   ==>  8   ==>2 4 6 8 10 12 14 16
                            2   ==>2 4
                    2  ==>  4   ==>2 4 6 8
                            2   ==>2 4
                            4   ==>2 4 6 8
                            6   ==>2 4 6 8 10 12
                    4  ==>  8   ==>2 4 6 8 10 12 14 16
                            2   ==>2 4
                            4   ==>2 4 6 8
                            6   ==>2 4 6 8 10 12
                            8   ==>2 4 6 8 10 12 14 16
                           10   ==>2 4 6 8 10 12 14 16 18 20
                    6  ==> 12   ==>2 4 6 8 10 12 14 16 18 20 22 24
                            2   ==>2 4
                            4   ==>2 4 6 8
                            6   ==>2 4 6 8 10 12
                            8   ==>2 4 6 8 10 12 14 16
                           10   ==>2 4 6 8 10 12 14 16 18 20
                           12   ==>2 4 6 8 10 12 14 16 18 20 22 24
                           14   ==>2 4 6 8 10 12 14 16 18 20 22 24 26 28
      2  ==>4  ==>8    ==> 16   ==>2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32
```

Figure 1: a_0 coefficient in sequences S_k^n of normal partitions

value) in vertical columns for n = 1 through 4, and horizontally (to conserve space) for n = 5. For n = 1 through 4, each entry a_0^n in S_i^n denotes to its right (with $\Rightarrow$) the *last* entry $< 2a_1, 0, \cdots, a_{n+1} >$ in the sequence $S_{a_0}^{n+1}$ that it generates. This figure simply reinforces the patterns emerging in the preceding computer enumerations.

Observe in this figure, that when n = 3, all 6 partitions with $a_0 \neq 0$ are enumerated in just two subsequences S_2^3 and S_4^3, which were generated by the two normal partitions in $\mathbf{P}^2$. With n = 4 the 26 normal partitions of $\mathbf{P}^4$ are enumerated in two occurrences of the subsequences S_2^4 and S_4^4, together with single occurrences of S_6^4 and S_8^4, which themselves were generated from the 6 normal partitions of $\mathbf{P}^3$. Fortunately, since all sequences S_k^n have the form 2, 4, $\cdots, k$, we need only keep track of the number of such sequences in $\mathbf{P}^n$, not their actual composition.

Let σ_k^n, k even, denote the *number* of subsequences S_k^n of normal partitions in $\mathbf{P}^n$. Based on Figure 1 we can construct Table 2.

n	2	3	4	5	6
y_n	2	6	26	166	1,626
k			σ_k^n		
2	1	1	2	6	26
4		1	2	6	26
6			1	4	20
8			1	4	20
10				2	14
12				2	14
14				1	10
16				1	10
18					6
20					6
22					4
24					4
26					2
28					2
30					1
32					1

Table 2: Counts σ_k^n of subsequences S_k^n of normal partitions in $\mathbf{P}^n$

Since every normal partition of $\mathbf{P}^n$ belongs to such a subsequence, we have

$$y_n = \sum_{even\ \ k}^{2^{n-1}} k \cdot \sigma_k^n \tag{3}$$

Using Table 2 and equation (3) one obtains $y_7 = 25,510$, and by (2) $z_7 = p_6 = 1,828$, so $p_7 = 27,338$. It only remains to determine $\sigma_k^{n+1}, 2 \leq k \leq 2^n$ given $\sigma_j^n, 2 \leq j \leq 2^{n-1}$.

Since each sequence S_k^{n-1} of normal partitions in $\mathbf{P}^{n-1}$ generates the subsequences $S_2^n, S_4^n, \cdots, S_{2k}^n$ in $\mathbf{P}^n$, one can simply loop over all such subsequences σ_k^{n-1} and increment $\sigma_2^n, \cdots \sigma_{2k}^n$ as in the following code section

```
max_k = 2**(n-1);
for (k=2; k<=max_k; k+=2)
        {
        for (j=2; j<=2*k; j+=2)
                sigma[n][j] += sigma[n-1][k];
        }
```

The $O(k^2)$ behavior of this double loop can become expensive when $k = 2^{n-1}$ becomes large. We observe in Table 2, that the first two values of σ_k^n are determined by

$$\sigma_2^n = \sigma_4^n = y_{n-2} \tag{4}$$

and that subsequent values of σ_k^n can be calculated as

$$\sigma_k^n = \sigma_{k+2}^n = \sigma_{k-2}^n - \sigma_{\lfloor (k+2)/2 \rfloor -2}^{n-1} \tag{5}$$

for $k = 6, 10, 14, \cdots$.

Putting together (2), (3), (4), and (5) one obtains

Theorem 2.1 *The number, p_n, of distinct partitions of 2^n is given by:*

$$p_n = p_{n-1} + \sum_{even k}^{2^{n-1}} k \cdot \sigma_k^n$$

where $\sigma_k^n = \begin{cases} \sum_{even\ i} \ k \cdot \sigma_i^{n-2} & : \quad k = 2, 4 \\ \sigma_{k-2}^n - \sigma_{\lfloor (k+2)/2 \rfloor -2}^{n-1} & : \quad k = (6,8), (10,12) \cdots \end{cases}$

The primary advantage of expressing p_n in this manner is that it permits the following counting procedure, which although somewhat more complex, has linear behavior.

```
long    sigma[MAX_N+1][POWER_MAX_N];

long    calculate_y (int n)
        /*
        ** Assumes sigma[n-1, 2**(n-2)] has been previously determined
        ** and globally stored.
        ** This procedure sets up sigma[n, 2**(n-1)], and returns
        ** the number y[n] of normal partitions with a[0] != 0
        */
        {
        int     k, k_calc, max_k;
        long    sum;

        max_k = 2**(n-1);
        switch (n)
          {
          case 1:
                return 1;
          case 2:
                sigma[2][2] = 1;
                break;
          case 3:
                sigma[3][2] = 1;
                sigma[3][4] = 1;
                break;
          default:
                sigma[n][2] = y[n-2];
```

```
            sigma[n][4] = y[n-2];

            for (k=6; k<=max_k; k+=4)
                    {
                    k_calc = (k+2)/2 - 2;
                    sigma[n][k]   = sigma[n][k-2] - sigma[n-1][k_calc];
                    sigma[n][k+2] = sigma[n][k-2] - sigma[n-1][k_calc];
                    }
            break;
   }
sum = 0;
for (k=2; k<=max_k; k += 2)
            {
            sum = sum + sigma[n][k]*k;
            }
y[n] = sum;
return sum;
}
```

With this code one can generate the following Table 3 of partitions of 2^n. The values of p_7

n	p_n	y_n	z_n
3	10	6	4
4	36	26	10
5	202	166	36
6	1,828	1,626	202
7	27,338	25,510	1,828
8	692,004	664,666	27,338
9	30,251,722	29,559,718	692,004
10	2,320,518,948	2,290,267,226	30,251,722

Table 3: Total p_n, normal y_n, and non-normal z_n partitions of 2^n

and p_8 have been verified by enumeration of all partitions, using the program of section 1.

We can get some sense of the growth of p_n by comparing it with other functions. In Table 4, we calculate p_n using floating point double arithmetic instead of the long integer arithmetic of Table 3 (which overflows with $n > 10$), and compare it multiplicatively with the function n^n. Based on this evidence, we offer without proof, the observation:

Theorem 2.2 *The number, $p_n = |\mathbf{P}^n|$, of distinct partitions of 2^n is bounded below by n^n for $n > 10$.*

n	p_n	n^n	p_n/n^n
2	$4.000\ 10^0$	$4.000\ 10^0$	$1.000\ 10^0$
3	$1.000\ 10^1$	$2.700\ 10^1$	$3.703\ 10^{-1}$
4	$3.600\ 10^1$	$2.560\ 10^2$	$1.406\ 10^{-1}$
5	$2.020\ 10^2$	$3.125\ 10^3$	$6.464\ 10^{-2}$
6	$1.828\ 10^3$	$4.665\ 10^4$	$3.918\ 10^{-2}$
7	$2.733\ 10^4$	$8.235\ 10^5$	$3.319\ 10^{-2}$
8	$6.920\ 10^5$	$1.677\ 10^7$	$4.125\ 10^{-2}$
9	$3.025\ 10^7$	$3.874\ 10^8$	$7.808\ 10^{-2}$
10	$2.321\ 10^9$	$1.000\ 10^{10}$	$2.321\ 10^{-1}$
11	$3.164\ 10^{11}$	$2.853\ 10^{11}$	$1.108\ 10^0$
12	$7.748\ 10^{13}$	$8.916\ 10^{12}$	$8.689\ 10^0$
13	$3.439\ 10^{16}$	$3.088\ 10^{14}$	$1.136\ 10^{+2}$
14	$2.789\ 10^{19}$	$1.111\ 10^{16}$	$2.510\ 10^{+3}$
15	$4.160\ 10^{22}$	$4.379\ 10^{17}$	$9.501\ 10^{+4}$

Table 4: Comparison of p_n with n^n

References

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