Transformations of Antimatroid Closure Spaces*

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Abstract

Investigation of the transformations of vector spaces, whose most abstract formulations are called *matroids*, is basic in mathematics; but transformations of discrete spaces have received relatively little attention. This paper develops the concept of transformations of discrete spaces in the context of *antimatroid closure spaces*. The nature of these transformations are quite different from those encountered in linear algebra because the underlying spaces are strikingly different. The transformation properties of "closed", "continuous" and "order preserving" are defined and explored. The classic graph transformations, homomorphism and topological sort, are examined in the context of these properties.

Then we define a *deletion* which we believe plays a central role in discrete transformations. Antimatroid closure spaces, when partially ordered, can be interpreted as lattices. We show that deletions induce lower semihomomorphisms between the corresponding lattices.

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1 Introduction

By a discrete space we mean a set of elements, points, or other phenomena which we will generically call our *universe*, denoted by $\mathbf{U}$. Individual points of $\mathbf{U}$ will be denoted by lower case letters: $a, b, ..., p, q, ... \in \mathbf{U}$. By $2^{\mathbf{U}}$, we mean the powerset on $\mathbf{U}$, or collection of all subsets of $\mathbf{U}$. Elements of $2^{\mathbf{U}}$ we will denote by upper case letters: S, T, X, Y, Z.

Throughout this paper we will be concerned with functions, or operators, defined on $2^{\mathbf{U}}$, in contrast to functions defined on $\mathbf{U}$. We call them **transformations**, and we will denote them using a postfix notation in which the operator/transformation symbol is separated from its preceding operand/argument by a *period*. We notationally conform to that found in [8] in which binary operations are denoted by infix expressions and unary operations are denoted by suffix expressions. For example, if $2^{\mathbf{U}} \xrightarrow{f} 2^{\mathbf{U}}$ is a transformation mapping subsets of $\mathbf{U}$ into other subsets of $\mathbf{U}$, then for $X, Y \subseteq \mathbf{U}$, we would use the expression $(X \cap Y).f$ to denote the image of their intersection.

One such transformation is a uniquely generated **closure** operator, φ, by which we mean a function $2^{\mathbf{U}} \xrightarrow{\varphi} 2^{\mathbf{U}}$ satisfying the closure axioms:

$X \subseteq X.\varphi$
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$
$X.\varphi.\varphi = X.\varphi^2 = X.\varphi$
$X.\varphi = Y.\varphi$ implies $(X \cap Y).\varphi = X.\varphi = Y.\varphi$

The last axiom is non-standard. It is not hard to show that closure operators which satisfy this additional axiom are uniquely generated in the sense that for any set Y, there exists a unique minimal set $X \subseteq Y$ such that $X.\varphi = Y.\varphi$. One can also show [16] that

Theorem 1.1 *A closure operator is uniquely generated if and only if it satisfies the anti-exchange property*

if $p, q \notin X.\varphi$ *then* $p \in (X \cup \{q\}).\varphi$ *implies* $q \notin (X \cup \{p\}).\varphi$.

In contrast, any set of elements $\mathbf{U}$ with an operator σ satisfying the first three closure axioms, together with the Steinitz-MacLane exchange axiom

if $p, q \notin X.\sigma$ then $p \in (X \cup \{q\}).\sigma$ implies $q \in (X \cup \{p\}).\sigma$

is called a *matroid* [10] [17] [2].[1] Any set $\mathbf{U}$ and closure operator φ satisfying an anti-exchange axiom, $(\mathbf{U}, \varphi)$, is called an **antimatroid** [3] [9], or **closure space** [16].[2] Other common names for this concept are *APS greedoid, shelling structure* [7], *alignment* [6], or *convex geometry* [5] provided only that one further requires the empty set, $\emptyset$, to be closed.

[1] The closure operator σ of a matroid is normally called the spanning operator.

[2] Closure spaces are far more abundant than one might expect. For example, there exist at least 202 distinct closure spaces comprised of 5 elements. More generally, it can be shown that there exist more than n^n distinct, non-isomorphic closure spaces provided $n \geq 10$ [15]. Similarly, there are many different closure operators, φ.

By the **generator** of X, or basis[3] of X, denoted $X.\beta$, we mean a minimal set Y such that $Y.\varphi = X.\varphi$. β is another well-defined transformation of $2^{\mathbf{U}}$ into $2^{\mathbf{U}}$.

The classical way of discussing the properties of functions is in terms of the topologies of their domains and codomains. We will use closure spaces associated with a discrete space, $\mathbf{U}$, in much the same way, as the underlying structure by which to discuss the properties of a transformation *f*.

Antimatroid closure spaces have been studied in [16], in which the subsets $X, Y \subseteq \mathbf{U}$ are partially ordered by $\leq_\varphi$ where,

$$X \leq_\varphi Y \qquad \text{if and only if} \qquad Y \cap X.\varphi \subseteq X \subseteq Y.\varphi \tag{1}$$

This is a partial order on *all* the subsets of $\mathbf{U}$, not just its closed subsets. It is possible to show that this partial ordering of $2^{\mathbf{U}}$ is, in fact, a well structured lattice, $\mathcal{L}$, called the **closure lattice** of $\mathbf{U}$. Figure 1(a) illustrates a typical closure lattice.
The regularity of structure suggested by this figure really exists, *c.f.* [16]. The collection of **closed** subsets, for which $X = X.\varphi$, forms a lower semimodular sublattice $[\emptyset, \mathbf{abcde}]$, denoted in this figure by bolder strings and joined by solid lines that are generally inclined from the lower left to the upper right which denote covering relationships.[4]

The generators, b, c, e, bc, be, d and de, are connected to the corresponding closed sets that they generate by dashed lines generally inclined from lower right to the upper left. It can be shown that each of the lattice intervals $[X.\varphi, X.\beta]$ is a boolean lattice. In the case of the 8 subsets comprising $[\mathbf{abcde}, de]$ and $[\mathbf{abcd}, d]$, we indicate their constituent elements and a dashed outline.

The dotted lines denote a few of the covering relationships between non-closed elements in different boolean intervals. These covering relationships, which we denote by $X \prec_\varphi Z$, do indeed echo those of the closed subgraph sublattice. In particular, we have the following results which can be found in [16].

Theorem 1.2 (Fundamental Covering Theorem) *If $p \notin X$ then*

(a) $X \leq_\varphi X \cup \{p\}$ *if and only if* $p \notin X.\varphi$
(b) $X \cup \{p\} \leq_\varphi X$ *if and only if* $p \in X.\varphi$

where (a) is a cover if and only if $(X \cup \{p\}).\varphi = X.\varphi \cup \{p\}$ *and*
(b) is always a covering relationship.

Moreover, if φ is uniquely generated then (a) and (b) characterize all covering relations in $(2^{\mathbf{U}}, \leq_\varphi)$.

[3]The term "basis" has so many connotations, especially with respect to vector spaces and their change of basis, that we prefer the more neutral "generator".

[4]The lower semimodularity of closed subsets partially ordered by inclusion has been repeatedly discovered by many authors. See Monjardet [11] for an interesting summary.

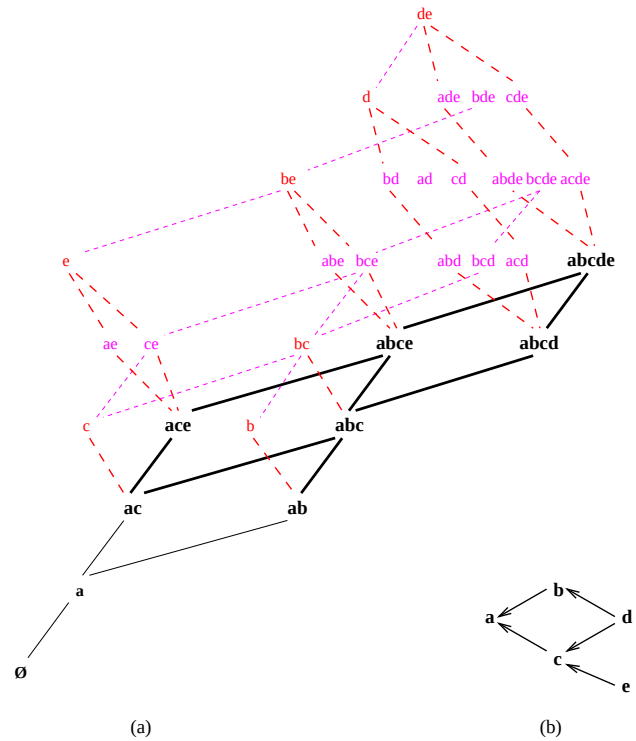


Figure 1: A Closure Lattice, $\mathcal{L}$

Corollary 1.3 *Let $Y \prec_\varphi Z$.*

(a) If $Z = Y \cup \{p\}$, then both Y and Z are closed.
(b) If $Z = Y - \{p\}$, then Z is not closed.

Lemma 1.4 *If φ is uniquely generated, and if $Z \neq \emptyset$ is closed,*

(a) $p \in Z.\beta$ if and only if $Z - \{p\}$ is closed,
in which case $Z.\beta - \{p\} \subseteq (Z - \{p\}).\beta$;
(b) $p, q \in Z.\beta$ implies there exist closed sets $Y_p, Y_q \subset Z$
such that $p \in Y_p, q \in Y_q$ and $p \notin Y_q, q \notin Y_p$;
(c) if $\emptyset.\varphi = \emptyset$, there exists $p \in Z$ such that $\{p\}$ is closed.

Theorem 1.5 (Fundamental Structure Theorem) *Let $X.\varphi \leq_\varphi Y.\varphi$ and let $X \in [X.\varphi, X.\beta]$. There exists a unique $Y \in [Y.\varphi, Y.\beta]$ such that $X \leq_\varphi Y$, where Y is minimal wrt. $\leq_\varphi$ (maximal wrt. $\subseteq$). Moreover $Y = X \cup \Delta$ where $\Delta = Y.\varphi - X.\varphi$ and $Y = Y.\varphi - \delta$ where $\delta = X.\varphi - X$.*

Figure 1(a) illustrates this theorem. Every interval $[X.\varphi, X.\beta]$ can be projected "upwards". By Theorem 1.2, every covering relation is marked by the difference of just one element between the two sets. Consequently, it can be illustrative to label covering relations (edges) with the corresponding element.

In our development of transformations we will also use the following lemmas, which have the feeling and flavor of relative topologies. Let $(\mathbf{U}, \varphi)$ be any closure space and let $W \subseteq \mathbf{U}$. By the **restriction of φ to W**, denoted $\varphi|_W$, we mean

$$Y.\varphi|_W = Y.\varphi \cap W, \quad \forall\, Y \subseteq W.$$

We will also call this a **relative closure**.

Lemma 1.6 *$\varphi|_W$ is a closure operator, which is uniquely generated if φ is.*

Proof:

(1) Because $X \subseteq W$, $X \subseteq X.\varphi \cap W = X.\varphi|_W$.

(2) Let $X \subseteq Y \subseteq W$. $X.\varphi|_W = X.\varphi \cap W \subseteq Y.\varphi \cap W = Y.\varphi|_W$.

(3)
$$\begin{aligned}(X.\varphi|_W).\varphi|_W &= (X.\varphi \cap W).\varphi|_W \\ &= (X.\varphi \cap W).\varphi \cap W \\ &\subseteq X.\varphi.\varphi \cap W.\varphi \cap W \\ &= X.\varphi \cap W = X.\varphi|_W\end{aligned}$$
The other containment follows from (1).

(4) Let φ be uniquely generated and let $X.\varphi|_W = Y.\varphi|_W$. $X.\varphi \cap W = Y.\varphi \cap W$ implies $X.\varphi = Y.\varphi$ (since $X.\varphi \subseteq W$ and $Y.\varphi \subseteq W$). So, $X.\varphi|_W = X.\varphi \cap W = (X \cap Y).\varphi \cap W = (X \cap Y).\varphi|_W$. □

Corollary 1.7 *Any subset W of an antimatroid closure space (U, φ) generates a corresponding subspace $(W, \varphi|_W)$.*

As shown below, the restriction of a closed set will always be closed. The ability to infer that X is closed wrt. φ when its restriction is closed wrt. $\varphi|_W$ is of more interest. The following lemma gives two sufficient conditions. Neither is necessary.

Lemma 1.8

(a) X closed wrt. φ implies $X \cap W$ is closed wrt. $\varphi|_W$.

(b) If W is closed wrt. φ and $X \subseteq W$, then
X closed wrt. $\varphi|_W$ implies X is closed wrt. φ.
(c) If X is closed wrt. $\varphi|_W$ and $(X.\varphi - X) \cap (\mathbf{U} - W) = \emptyset$, then
X closed wrt. φ.

Proof:

(a) If X is closed wrt. φ then $X.\varphi = X$ and
$(X \cap W).\varphi|_W \subseteq X.\varphi|_w \cap W.\varphi|_W = X.\varphi \cap W \cap W = X \cap W$.

(b) X closed wrt. $\varphi|_W$ implies $X.\varphi \cap W = X$.
But $X \subseteq W$ and W closed imply $X.\varphi \subseteq W$, so $X.\varphi \cap W = X.\varphi = X$.

(c) If X is not closed wrt. φ, then $(X.\varphi - X) = \Delta$.
Let X be closed wrt. $\varphi|_W$ so $\emptyset = X.\varphi|_W - X = (X.\varphi \cap W) - X = (X.\varphi - X) \cap W$. Consequently $\Delta \subseteq \mathbf{U} - W$, and $(X.\varphi - X) \cap (\mathbf{U} - W) \neq \emptyset$. $\square$

2 Transformations

By a **transformation** of $\mathbf{U}$ to $\mathbf{U}'$, we mean a function f which maps $2^{\mathbf{U}}$ into $2^{\mathbf{U}'}$, and which we usually denote in terms of the base sets as $\mathbf{U} \xrightarrow{f} \mathbf{U}'$.

We have already seen two transformations, $\mathbf{U} \xrightarrow{\varphi} \mathbf{U}$ and $\mathbf{U} \xrightarrow{\beta} \mathbf{U}$ which map the sets of $2^{\mathbf{U}}$ into $2^{\mathbf{U}}$. Another well-known transformation is the **natural extension** of a point map $f : \mathbf{U} \to \mathbf{U}'$ to subsets of $\mathbf{U}$ using the familiar definition $Y.f = \{y' \in \mathbf{U}' | (\exists y \in Y)[y' = y.f]\}$. Still two more transformations are the upper, and lower, bound operators on a partially ordered set.

To ground our intuitive understanding of transformations in terms of the natural extension of a point map (as is customary in topology) is to invite confusion. It is much better to begin with a closure operator, φ, as one's paradigm because transformations can be wildly misbehaved. In fact, probably the best definition of *chaos* is given in terms of transformations [1]. There is ample scope for misbehavior because we observe that if $\mathbf{U}$ and $\mathbf{U}'$ are sets of n elements each, there exist only n^n distinct functions $f : \mathbf{U} \to \mathbf{U}'$ compared to $(2^n)^{2^n}$ transformations $\mathbf{U} \xrightarrow{f} \mathbf{U}'$. To achieve any results of interest we must constrain the transformations.

One can enumerate a long, non-exhaustive list of transformation properties, *e.g.* f is said to be:

contractive	if $\lvert Y.f\rvert \leq \lvert Y\rvert$;
expansive	if $\lvert Y.f\rvert \geq \lvert Y\rvert$;
monotone	if $X \subseteq Y$ implies $X.f \subseteq Y.f$;
antimonotone	if $X \subseteq Y$ implies $X.f \supseteq Y.f$;

uniquely generated	if $X.f = Y.f$ implies $(X \cap Y).f = X.f$,
union preserving	if $(X \cup Y).f = X.f \cup Y.f$; or
stable	if $Y.f \subseteq Y$,

for all $X, Y \subseteq \mathbf{U}$.

We observe that closure, φ, is monotone and expansive; while β is contractive and stable, but not monotone. φ need not be uniquely generated, but the results of this paper are restricted to examples where it is. Lower, and upper, bound operators are antimonotone; and the natural extension of a point function is union preserving and monotone.

If we regard f as a mapping defined on the closure space $(\mathbf{U}, \varphi)$, as suggested by Figure 2, then we may begin to explore other properties associated with f. Foremost is the question,

$$\begin{tikzcd}
\mathbf{U} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbf{U}' \arrow[d, "\varphi'"] \\
(\mathbf{U}, \varphi) \arrow[r, "f"] & (\mathbf{U}', \varphi')
\end{tikzcd}$$

Figure 2: f regarded as a closure space transformation

when is the diagram of Figure 2 commutative?

2.1 Continuous and Closed Transformations

Any transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ that maps closed sets of $\mathbf{U}$ into the closed sets of $\mathbf{U}'$ would naturally be called a **closed** transformation by analogy to topologically open maps. That is, f is closed if Y closed in $(\mathbf{U}, \varphi)$ implies $Y.f$ is closed in $(\mathbf{U}.f, \varphi')$. In these definitions we tacitly assume that $\mathbf{U}.f$ is closed in $\mathbf{U}'$ with respect to φ'. This ensures, by Lemma 1.8, that the relative closure of φ' with respect to $\mathbf{U}.f$ is conformable with the closure on $\mathbf{U}'$.

Lemma 2.1 *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is monotone and closed, then*

$$X.f.\varphi' \subseteq X.\varphi.f, \qquad \forall X \subseteq \mathbf{U}$$

Proof: By monotonicity, $X.f \subseteq X.\varphi.f$. But, since $X.\varphi$ is closed and f is closed, $X.f.\varphi' \subseteq X.\varphi.f$. □

Monotonicity appears to be a basic property. It can be used to obtain several interesting results. Occasionally, we also want a weak form of inverse monotonicity; that is, we want

to be able to assert that there exists at least one pre-image set also satisfying the inclusion property. A transformation $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ is said to be **upper monotone** if

(a) it is monotone, *i.e.* $X \subseteq Y$ implies $X.f \subseteq Y.f$, and
(b) $X.f \subseteq Y.f$ implies there exists Y_0 such that $Y_0.f = Y.f$ and $X \subseteq Y_0$.

It is easily shown that:

Lemma 2.2 *Let $U \xrightarrow{f} U'$ be any transformation.*

(a) If $(X \cap Y).f = X.f \cap Y.f$, then f is monotone.

(b) If $(X \cup Y).f = X.f \cup Y.f$, then f is upper monotone

Proof:

(a) $X \subseteq Y$ implies $X \cap Y = X$. Consequently, $X.f \cap Y.f = (X \cap Y).f = X.f$, implying $X.f \subseteq Y.f$.

(b) Let $X \subseteq Y$ implying $X \cup Y = Y$. Then, $X.f \cup Y.f = (X \cup Y).f = Y.f$ implying $X.f \subseteq Y.f$. To establish upper monotonicity, let $X.f \subseteq Y.f$. Let $Y_0 = X \cup Y$. Then $Y_0.f = (X \cup Y).f = X.f \cup Y.f = Y.f$ and $X \subseteq Y_0$. $\square$

A transformation $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ is said to be **continuous** if $Y.f$ closed in $(\mathbf{U}.f, \varphi')$ implies $Y.\varphi.f = Y.f$

Continuity is traditionally defined in terms of properties of the inverse map, as in "the inverse image of closed sets is closed". This appears different. Readily, $Y'.f^{-1}$ is a collection of sets $\{Y \subseteq \mathbf{U} | Y.f = Y'\}$. To be continuous every set $Y \in Y'.f^{-1}$ need not be closed in $\mathbf{U}$ just because Y' is closed in $\mathbf{U}'$; but whenever $Y \in Y'.f^{-1}$ its closure must be as well. For an equivalent formulation, let Y' be closed in $\mathbf{U}'$ and let $Y'.f^{-1}$ be partially ordered by inclusion. If f is continuous, then Y maximal in $Y'.f^{-1}$ implies Y is closed in $\mathbf{U}$.

To understand the reason for this definition, consider the simple transformation f of Figure 3 which maps a linear order on 3 points onto a linear order on 2 points, as indicated by the assignments to the right. This f could be regarded as an epitome of a "continuous" graph transformation. The graph on $\{a', c'\}$ is closed with respect to any of the closure operators that we normally associate with such acyclic graphs, while $\{ac\} \in \{a', c'\}.f^{-1}$ is closed with respect to none of them. Clearly, we can't require every pre-image to be closed. We believe the definition of "continuity" that we have given captures the intuitive notion of the concept correctly. The second motivation is Lemma 2.3, below.

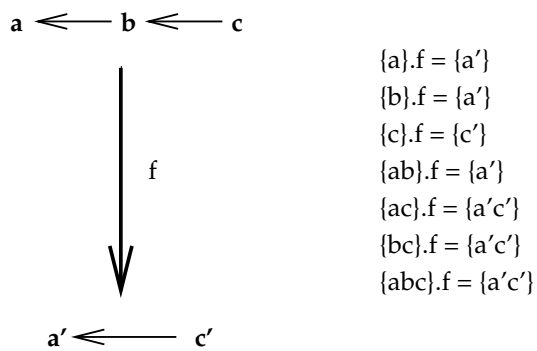


Figure 3: A "continuous" transformation

Lemma 2.3 *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is onto, upper monotone and continuous with* $\mathbf{U}.f$ *closed in* $(\mathbf{U}', \varphi')$ *then*

$$Y.\varphi.f \subseteq Y.f.\varphi', \quad \forall Y \subseteq U$$

Proof: $Y.f \subseteq Y.f.\varphi'$. Let $Z' = Y.f.\varphi' \subseteq \mathbf{U}.f$. Since f is onto, $\exists Z$ such that $Z.f = Z'$. Now by upper monotonicity, $\exists Z_0, Y \subseteq Z_0$ such that $Z_0.f = Z.f = Y.f.\varphi'$. Finally, since f is continuous, $Z_0.\varphi.f = Y.f.\varphi'$. But, since $Y \subseteq Z_0$, $Y.\varphi \subseteq Z_0.\varphi$, and by monotonicity, $Y.\varphi.f \subseteq Z_0.\varphi.f = Y.f.\varphi'$. □

Corollary 2.4 *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is onto, upper monotone, continuous, and closed then*

$$X.\varphi.f = X.f.\varphi', \quad \forall X \subseteq \mathbf{U}$$

which establishes sufficient conditions for φ and f to be commutative transformations in Figure 2.

2.2 Transformations of Graph Theory

Graph theory provides a rich load of transformations over discrete spaces, particularly graph homomorphisms. In this section we describe graph homomorphisms in terms of the properties we have just defined. Of particular interest will be the transformation properties of monotonicity, continuity and closure.

Let $G = (N, E)$. One may regard either the nodes, points, or vertices, N, of the graph as one's universe, $\mathbf{U}$; or one may treat the edge set, E, as $\mathbf{U}$. For this paper, we assume the former. Let $G' = (N', E')$. A function $f : N \to N'$ is called a **graph homomorphism**[5] , denoted $G \xrightarrow{f} G'$, if

[5]Part (b) of this definition requires a preimage under f for every edge in G'. Some authors do not impose this requirement.

(a) $(x, y) \in E$ and $x.f \neq y.f$ imply $(x.f, y.f) \in E'$
(b) $(x', y') \in E'$ implies there exists $(x, y) \in E$ such that
$$x.f = x' \text{ and } y.f = y'.$$

The transformation of Figure 3 is a graph homomorphism.

Since it is not hard to show that transitive closure on an edge set E is uniquely generated if and only if G is acyclic, we assume N is partially ordered by $\leq$, and we assume that φ is one of

$$\begin{aligned} Y.\varphi_L &= \{x \mid x \leq y, \quad y \in Y\} \\ Y.\varphi_R &= \{z \mid y \leq z, \quad y \in Y\} \\ Y.\varphi_C &= \{x \mid y_1 \leq x \leq y_2, \quad y_1, y_2 \in Y\}. \end{aligned} \tag{2}$$

which we collectively call **path closures**. The first two are *ideal* operators, the latter is an *interval* operator.[6] The closure of Figure 1(a) is obtained by applying φ_L to the graph of Figure 1(b).

By a **subgraph** H of G we mean any set $N_H \subseteq N$ with edge set $E|_{N_H}$, that is a *full* subgraph.

Lemma 2.5 *Let $f : G \to G'$ be a graph homomorphism, and let φ be any path closure. If H' is closed in G' then $H = H'.f^{-1}$ is closed in G.*

Proof: Readily $x \leq z$ implies $x.f \leq z.f$.
case φ_L: Let H' be closed in G' so that $z' \in H'$, $x' \leq z'$ imply $x' \in H'$. Let $z \in H'.f^{-1} \subseteq \mathbf{U}'$ and let $x \leq z$. Since f is a homomorphism $x.f \leq' z.f$. Because H' is left closed, $x.f \in H'$ implying $x \in H'.f^{-1}$ so $H'.f^{-1}$ is left closed.
case φ_R: Virtually identical.
case φ_C: Let H' be closed wrt. φ_C, and let $x, z \subseteq H'.f^{-1}$. If y is a point on a path, such that $x \leq y \leq z$ then $y.f$ is on a path $x.f \leq y.f \leq z.f$ in G'. Since H' is φ_C closed, $y.f \in H'$ implying $y \in H'.f^{-1}$ which is therefore also closed wrt. φ_C. $\square$

We have used the notation, $f : G \to G'$ to emphasize that f regarded as a graph homomorphism is simply a function mapping one node set into another; it is *not* really a transformation. The *natural extension*, in which $Y.f = \{y' \in N' | (\exists y \in Y)[y' = y.f]\}$ makes it a transformation of 2^N to $2^{N'}$.

Lemma 2.6 *Let $f : G \to G'$ be a graph homomorphism, and let φ be any path closure. The natural extension $G \xrightarrow{f} G'$ is continuous.*

Proof: Readily, any natural extension f preserves unions, so by 2.2 is upper monotone. Let H be a subgraph of G such that $H.f$ is closed. So $H \subseteq H.f.f^{-1}$, which is the set of all points $x \in P$

[6] In [13], the author called the interval operator the *convex hull* operator; in [5] [6], it is called an *order convex* operator.

such that $x.f \in H.f$. By Lemma 2.5, $H.f.f^{-1}$ is closed. Consequently, $H \subseteq H.\varphi \subseteq H.f.f^{-1}$ implies $H.f \subseteq H.\varphi.f \subseteq H.f$; so equality follows. $\square$

While the natural extension of any graph homomorphism f must be continuous, it need not be closed as the counter example of Figure 4 illustrates. Readily, the set $\{abd\}$ is closed

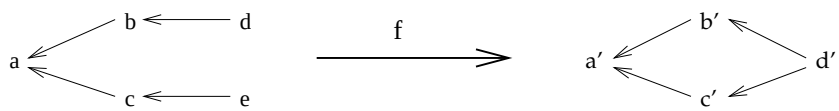


Figure 4: Graph homomorphism, f, that is not closed

in $\mathbf{U}$ with respect to φ_L, but its image $\{a'b'd'\}$ is not closed in $\mathbf{U}'$ because $c' \in \{a'b'd'\}.\varphi_L$.

The topological sort, τ, of a partially ordered set provides another example. It is easy to show that τ is not normally closed with respect to either φ_L or φ_C; but that it is continuous, onto, and upper monotone. Indeed, since $Y.\varphi_L = \{x | x < y, y \in Y\}$, the fact that $Y.\varphi_L.\tau \subseteq Y.\tau.\varphi_L$, or equivalently that $x \leq y$ implies $x.\tau \leq y.\tau$ is often taken to be its definition.

It is not surprising that graph homomorphisms and topological sorts are continuous. Just as in analysis, it is natural to first define and study well-behaved transformations.

2.3 Order Preserving Transformations

Equation (1) defines partial orders $\leq_\varphi$ and $\leq_{\varphi'}$ on $(\mathbf{U}, \varphi)$ and $(\mathbf{U}', \varphi')$ respectively. A transformation f is said to be **order preserving** if $X \leq_\varphi Y$ implies $X.f \leq_{\varphi'} Y.f$ for all $X, Y \subseteq \mathbf{U}$. It would be hoped that closed, continuous transformations would be order preserving. But, this need not be true. Consider Figure 5, in which the nodes a and b of the graph on the left both map onto a' on the right. It is not hard to verify that f

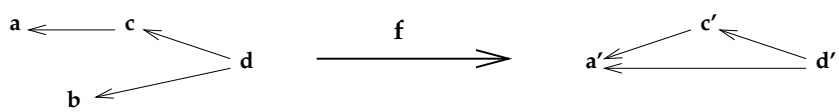


Figure 5: Graph homomorphism, f, that is not order preserving

is onto, upper monotonic, closed and continuous with respect to φ_L. But f is not order preserving w.r.t. φ_L, because if $X = \{c\}$ and $Y = \{bcd\}$, then readily $X \leq_\varphi Y$ because $Y \cap X.\varphi = \{c\} \subseteq X \subseteq Y.\varphi$, yet $X.f \not\leq_{\varphi'} Y.f$ because $Y.f \cap X.f.\varphi' = \{a'c'\} \not\subseteq \{c'\} = X.f$.

In Figure 5, f is a graph homomorphism, hence union preserving. So even adding this property to continuity, closure, onto and upper monotonicity is insufficient to ensure that f be order preserving. But,

Theorem 2.7 *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is closed, continuous, onto, upper monotone and intersection preserving, then* f *is order preserving.*

Proof: Let $X \leq_\varphi Y$, so $Y \cap X.\varphi \subseteq X \subseteq Y.\varphi$. Because f is intersection preserving, we have using Lemma 2.1, $Y.f \cap X.f.\varphi' \subseteq Y.f \cap X.\varphi.f = (Y \cap X.\varphi).f \subseteq X.f$. And, by Lemma 2.3, $X.f \subseteq Y.\varphi.f \subseteq Y.f.\varphi'$. or $X.f \leq_{\varphi'} Y.f$. $\square$

Closure and continuity, by themselves are insufficient to ensure that a transformation is order preserving; but one might hope that order preserving transformations are closed, continuous, and commute with φ, as in Figure 2. Unfortunately, we can only prove

Lemma 2.8 *If* $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ *is order preserving, then*

(a) f *is closed (provided* $\mathbf{U}.f$ *is closed in* $\mathbf{U}'$*),*
(b) $X.\varphi.f \subseteq X.f.\varphi'$, $\forall X \subseteq \mathbf{U}$.

Proof:

(a) Let X be closed in $(\mathbf{U}, \varphi)$, then $X \leq_\varphi \mathbf{U}$ implying $X.f \leq_{\varphi'} \mathbf{U}.f$, so $X.f$ is closed.

(b) First, $X.\varphi.f \subseteq X.f.\varphi'$ because $X \leq_\varphi X.\varphi$ implying $X.\varphi.f \leq_{\varphi'} X.f$
or, $X.f \cap X.\varphi.f.\varphi' \subseteq X.\varphi.f \subseteq X.f.\varphi'$. $\square$

But, if we add just the property of monotonicity we obtain

Corollary 2.9 *If* $(\mathbf{U}, \varphi) \xrightarrow{f} (\mathbf{U}', \varphi')$ *is monotone and order preserving then*

$$X.\varphi.f = X.f.\varphi', \quad \forall X \subseteq \mathbf{U}$$

Proof: By monotonicity, $X \subseteq X.\varphi$ implies $X.f \subseteq X.\varphi.f$. So because f is closed (Lemma 2.8), $X.f.\varphi' \subseteq X.\varphi.f.\varphi' = X.\varphi.f$. $\square$

In all the results of this section, monotonicity has been a key property. It is a kind of well-formed behavior that seems natural to expect. But making it so central may be wrong. The fundamental issue remains "when is $X.\varphi.f = X.f.\varphi'$?", but in the next section we approach it rather differently.

3 Deletions

Perhaps the simplest transformation that can be applied to a closure space is just the removal of one, or more, of its points. We call them deletions. But before defining deletions themselves it is convenient to develop a concept of dependence.

3.1 Dependence

We say that q is **dependent** on p with respect to Y if

$$q \notin Y.\varphi \quad \text{but} \quad q \in (Y \cup \{p\}).\varphi \ .$$

Readily $p \notin Y.\varphi$ else $(Y \cup \{p\}).\varphi = Y.\varphi$. Hence, by anti-exchange $p \notin (Y \cup \{q\}).\varphi$.

We let $Y.\Delta_p$ denote the set of all such q that are dependent on p with respect to Y. Clearly, if $p \notin Y.\varphi$ then $p \in Y.\Delta_p$. This, and other simple properties of these dependence sets, are summarized in:

Lemma 3.1 *In a uniquely generated, antimatroid closure space,* $(\mathbf{U}, \varphi)$,

(a) $\forall p, \quad X.\varphi \cap X.\Delta_p = \emptyset$.

(b) $p \in X.\varphi$ *implies* $X.\Delta_p = \emptyset$.

(c) $X.\Delta_p \neq \emptyset$ *implies* $p \in X.\Delta_p$.

(d) $X.\Delta_p = \{p\}$ *implies*

(1) $X \prec_\varphi X \cup \{p\}$,

(2) both X *and* $X \cup \{p\}$ *are closed.*

(e) $p \notin X.\varphi$ *and* $X \not\prec_\varphi X \cup \{p\}$ *if and only if* $\exists q \in X.\Delta_p, \quad q \neq p$, *(i.e.* $|X.\Delta_p| \geq 2$*).*

(f) $q \in X.\Delta_p, \ q \neq p$ *implies* $\emptyset \subset X.\Delta_q \subset X.\Delta_p$.

(g) $X.\Delta_p = X.\varphi.\Delta_p$

Proof:

(a)-(c) By definition.

(d) Corollary of Fundamental Covering Theorem 1.2(a).

(e) $p \notin X.\varphi$ implies $X \leq_\varphi X \cup \{p\}$ (by FCT 1.2), and since $X \not\prec_\varphi X \cup \{p\}$, $X.\varphi \cup \{p\} \subset (X \cup \{p\}).\varphi$ or $\exists q \in (X \cup \{p\}).\varphi, q \notin X.\varphi$ and $q \neq p$.

(f) By anti symmetry, $p \notin X.\Delta_q$.

(g) $q \in X.\Delta_p$ if and only if $q \notin X.\varphi$ and $q \in X \cup \{p\}.\Delta_p$ and in turn if and only if $q \in X.\varphi.\Delta_p$. □

While we define dependence with respect to arbitrary Y, by Lemma 3.1(g) we can restrict it to closed sets. In this case, the K-factor relation of [5] can be restated as

$$q \leq_Y p \ \equiv_{def} \ q \in Y.\Delta_p$$

and their Theorem 2.3, that $(\mathbf{U}, \varphi)$ is a *convex geometry* (*i.e.* uniquely generated) if and only if the relation $\leq_Y$ is a partial order for all closed Y, follows easily. Indeed, convex geometries provide a fertile source of antimatroid closure spaces. A simple 5 point geometry is illustrated in Figure 6(a). Its closure lattice, where the closure operator φ is the convex hull operator, is shown is Figure 6(b). Notice that $q \in \{abc\}.\Delta_p \subseteq \{ab\}.\Delta_p$ and $c \in \{aq\}.\Delta_p = \{abq\}.\Delta_p = \{cp\}$.

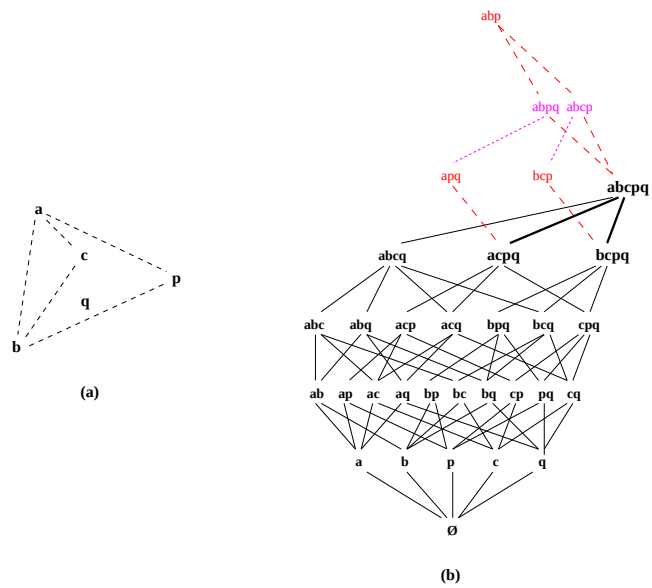


Figure 6: A convex geometry $(\mathbf{U}, \varphi)$ and $\mathcal{L}_{(U,\varphi)}$

Covering relationships are important in discrete structures, *e.g.* Theorem 1.2. In addition to the covering properties expressed in Lemma 3.1(d) and (e), we have

Lemma 3.2 *For all $X \subseteq \mathbf{U}$, if $p \notin X.\varphi$ then $X \cup X.\Delta_p$ covers $(X \cup X.\Delta_p) - \{p\}$.*

Proof: This is virtually a corollary of Lemma 3.1(d) since $X \cup X.\Delta_p = ((X \cup X.\Delta_p) - \{p\}).\Delta_p$ $\square$

Dependence is directly related to the lattice structure described in Theorem 1.5 which asserts if $Y.\varphi \leq_\varphi Z.\varphi$ then for each $Y \in [Y.\varphi, Y.\beta]$ there exists a unique $Z \in [Z.\varphi, Z.\beta]$ such that $Y \cup \Delta = Z$. Δ in this theorem is a dependence set. Paths from the lower left to upper right constitute dependence sets. Dependence, Δ, and generators, β, are intertwined by the following lemma.

Lemma 3.3 *For all $p \notin X.\varphi$, $\quad p \in (X \cup \{p\}).\beta \subseteq X.\beta \cup \{p\}$.*

Proof: Readily $(X \cup \{p\}).\beta \not\subseteq X.\varphi$, since $p \notin X.\varphi$. Let, $q \in (X \cup \{p\}).\beta$. $(X \cup \{q\} \cup \{p\}).\varphi = (X \cup \{p\}).\varphi$. Hence by unique generation property, $(X \cup \{p\}).\beta \subseteq (X \cup \{p\}) \cap (X \cup \{q\} \cup \{p\}) = X \cup \{p\}$ implying either $q = p$ or $q \in X$.
Since $p \notin X.\varphi, (X \cup \{p\}).\beta \not\subseteq X.\varphi$. So from the observation above, $p \in (X \cup \{p\}).\beta$.
Finally, by Lemma 1.4(a), $(X \cup \{p\}).\beta - \{p\} \subseteq X.\beta$. $\square$

The implications of this lemma can be seen in Figure 1(a). Since $\{e\}.\varphi_L = \{ace\} =$

$\{ce\}.\varphi_L$, $\{ce\}.\beta = \{e\}$. $\{ce\}.\Delta_b = \{b\}$ and $(\{ce\} \cup \{b\}).\beta = \{be\} = \{ce\}.\beta \cup \{b\}$. But, the same properties hold even when the dependency set is not a singleton. For example, $(\{ce\} \cup \{d\}).\beta = \{de\} = \{ce\}.\beta \cup \{d\}$.

3.2 The Deletion Transformation

By a **single point deletion**, χ_p, of a closure space $(\mathbf{U}, \varphi)$, we mean a transformation $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$, where $\mathbf{U}' = \mathbf{U} - \{p\}$ and $\varphi' = \varphi|_{U'}$ and χ_p is defined by:

$$Y.\chi_p = \begin{cases} Y & \text{if } p \notin Y \\ Y \cup (Y - \{p\}).\Delta_p - \{p\} & \text{otherwise,} \end{cases} \tag{3}$$

We observe, that χ_p is a well-defined, onto function, and that for notational clarity we may omit the subscript p when it is clear from the context what point is being eliminated in the deletion.

Lemma 3.4 $Y - \{p\} \subseteq Y.\sigma_p \subseteq Y.\varphi - \{p\} = Y.\varphi.\sigma_p, \quad \forall p \in \mathbf{U}, Y \subseteq \mathbf{U}$

Proof: If $p \notin Y$ the result is essentially trivial. Let $p \in Y$. We first observe that either $Y.\varphi$ covers, or is covered by, $Y.\varphi - \{p\}$, so $Y.\varphi.\chi_p = Y.\varphi - \{p\}$. Since $p \in Y$, $Y.\sigma_p = Y \cup (Y - \{p\}).\Delta_p - \{p\}$. But, since $(Y - \{p\}).\Delta_p = \{\, q \mid q \notin (Y - \{p\}).\varphi, q \in Y.\varphi \,\} \subseteq Y.\varphi$, it follows that $Y.\sigma_p \subseteq Y.\varphi - \{p\} = Y.\varphi.\chi_p$. □

Covering relationships involving the deleted element p are important. The impact of deletion is clearly revealed by the following definition which is equivalent to (3) and which is easily established using Theorem 1.2 and Lemmas 3.1 and 3.2.

$$Y.\chi_p = \begin{cases} Y & \text{if } p \notin Y \\ Y - \{p\} & \text{if } Y \text{ covers } Y - \{p\}, \text{ or } Y - \{p\} \text{ covers } Y \\ Y \cup (Y - \{p\}).\Delta_p - \{p\} & \text{otherwise,} \end{cases}$$

If Y is closed and $p \in Y$, then either $p \in Y.\beta$ whence Y covers $Y - \{p\}$ or $p \notin Y.\beta$ so $Y - \{p\}$ covers Y. In any case, Y closed implies $Y.\chi_p = Y - \{p\}$.

3.3 Properties of Deletion

In this section, we explore the transformation properties of deletion. For example, one would wish that deletions χ_p were monotone so the results of Section 2.1 could be applied; but they are not. Consider the single point deletion χ_d of Figure 7. Here, we use convex path closure, φ_C. The subsets of $(\mathbf{U}, \varphi_C)$ have been partitioned into pre-image subsets as indicated by dotted lines. The subsets Y of $(\mathbf{U}, \varphi_C)$ have been ordered by (1) to form $\mathcal{L}_{(U,\varphi_C)}$. Elements of $(\mathbf{U}', \varphi'_C)$ have the same labels, but primed, and are similarly ordered.

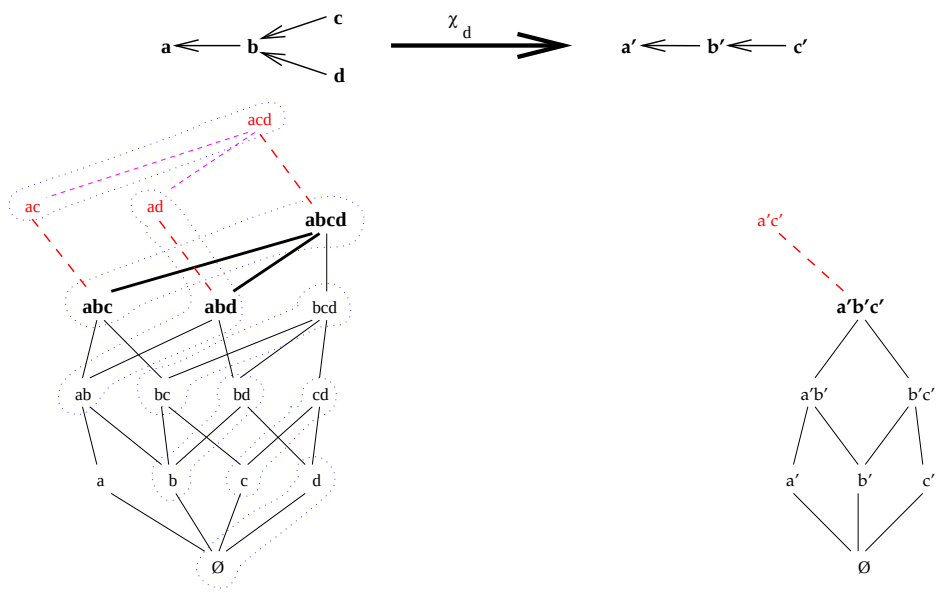


Figure 7: A single point deletion, χ_d, that is not monotone

Readily, $\{ad\} \subseteq \{acd\}$, but $\{ad\}.\chi_d = \{a'b'\} \not\subseteq \{a'c'\} = \{acd\}.\chi_d$. We can, however, prove a restricted form of monotonicity.

Lemma 3.5 *Let* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *be a single point deletion*

(a) If Y_1 *and* Y_2 *are closed, then* $(Y_1 \cap Y_2).\chi_p = Y_1.\chi_p \cap Y_2.\chi_p$.

Let $Y_1 \subseteq Y_2$.

(b) If either Y_1 *or* Y_2 *is closed, then* $Y_1.\chi_p \subseteq Y_2.\chi_p$.

(c) If $Y_1.\varphi = Y_2.\varphi$*, then* $Y_1.\chi_p \subseteq Y_2.\chi_p$.

Proof:

(a) Since Y_1 and Y_2 are closed, $Y_1 \cap Y_2$ must be as well. If $p \in Y_1$ (or Y_2 or $Y_1 \cap Y_2$) then $Y_1.\chi_p = Y_1 - \{p\}$. Consequently, $(Y_1 \cap Y_2).\chi_p = Y_1 \cap Y_2 - \{p\} = Y_1.\chi_p \cap Y_2.\chi_p$.

(b) Let Y_1 be closed. If $p \notin Y_1$ the result follows immediately. Otherwise, by Lemma 3.4, $Y_1.\chi_p = Y_1 - \{p\}$. So $Y_1.\chi_p = Y_1 - \{p\} \subseteq Y_2 - \{p\} \subseteq Y_2.\chi_p$.
Let Y_2 be closed. If $p \notin Y_2$, the result is again immediate. By Lemma 3.4, $Y_1.\chi_p \subseteq Y_1.\varphi - \{p\}$. But $Y_1 \subseteq Y_2$ implies $Y_1.\varphi \subseteq Y_2.\varphi = Y_2$, so $Y_1.\chi_p \subseteq Y_2.\varphi - \{p\} = Y_2.\chi_p$.

(c) $Y_1, Y_2 \in [Y.\varphi, Y.\beta]$. If $p \notin Y_2$, then $p \notin Y_1$, and $Y_1.\chi_p = Y_1 \subseteq Y_2 = Y_2.\chi_p$
If $p \in Y_2$, but $p \notin Y_1$, then $p \notin Y.\beta$ and $Y_2 - \{p\}$ covers Y_2. Consequently $Y_1.\chi_p = Y_1 \subseteq Y_2 - \{p\} = Y_2.\chi_p$.
Let $p \in Y_1$ implying $p \in Y_2$. If $p \notin Y.\beta$, then $Y_1 \prec_\varphi Y_1 - \{p\}$ and $Y_2 \prec_\varphi Y_2 - \{p\}$. Thus,

$Y_1.\chi_p = Y_1 - \{p\} \subseteq Y_2 - \{p\} = Y_2.\chi_p$. While if $p \in Y.\beta$, then $Y_1.\chi_p = Y_1 \cup (Y_1 - \{p\}).\Delta_p - \{p\} = Y_1 \cup \{\, q \mid q \in (Y_1 - \{p\}).\Delta_p \wedge q \in Y_1.\varphi\} - \{p\} \subseteq Y_2 \cup \{\, q \mid q \in (Y_2 - \{p\}).\Delta_p \wedge q \in Y_2.\varphi\} - \{p\} = Y_2.\chi_p$, because $Y_1.\varphi = Y_2.\varphi$. $\square$

Observe that if both Y_1 and Y_2 are closed, then Lemma 3.5(b) is a simple corollary of the preceding part (a) and Lemma 2.2.

Three more deletions, all defined on the 5 point graph of Figure 4, will help develop the sense of this transformation. Again, in all cases we use convex path closure, φ_C. The deletion, χ_c, of Figure 8 is the simplest. All sets are mapped onto themselves or to one they

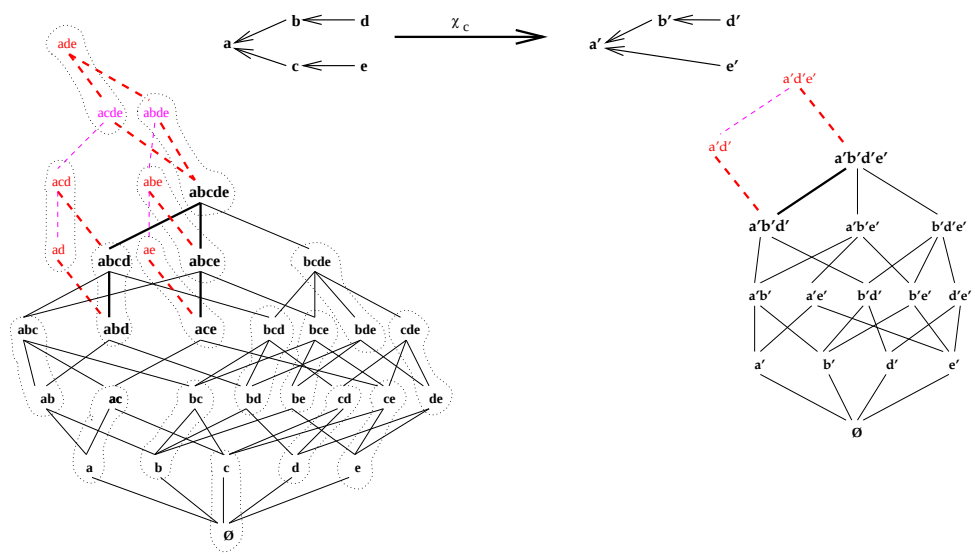


Figure 8: The single point deletion, χ_c applied to the closure space of Figure 4

cover (or are covered by). Notice however, that (a), $\{acde\}.\chi_c = \{a'd'e'\} \subset \{a'b'd'e'\} = \{abcde\}.\chi_c$, so Lemma 3.4 cannot be strengthened, and (b) that the closed set $\{ace\}$ is mapped onto $\{a'e'\}$ which is a generator in $(\mathbf{U}, \varphi_C)$, but closed in $(\mathbf{U}', \varphi'_C)$.

Lemma 3.6 *Single point deletions* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *are closed. I.e. if* Y *is closed wrt.* φ*, then* $Y.\chi_p$ *is closed wrt.* φ'*.*

Proof: Let Y be closed wrt φ. If $p \notin Y$, then by Lemma 1.8(a) $Y = Y.\chi_p$ is closed wrt. $\varphi|_U$. If $p \in Y$, then because Y is closed, either Y covers $Y - \{p\}(p \in Y.\beta)$, or *vice versa*. In either case, $Y.\chi_p = Y - \{p\}$. And since $Y.\chi_p = Y - \{p\} = Y \cap \mathbf{U}'$, $Y.\chi_p$ is closed wrt. $\varphi|_{U-\{p\}} = \varphi'$. $\square$

Corollary 3.7 corresponds to Lemma 2.1 except that it does not invoke monotonicity.

Corollary 3.7 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion then*

$$Y.\chi_p.\varphi' \subseteq Y.\varphi.\chi_p$$

Proof: By Lemma 3.4, $Y.\chi_p \subseteq Y.\varphi.\chi_p$. $Y.\varphi$ is closed, so because χ_p is closed (Lemma 3.6) $Y.\chi_p.\varphi' \subseteq Y.\varphi.\chi_p$. □

The deletion, χ_e, of Figure 9 is a bit more complex. For example, $\{ade\}$ maps onto

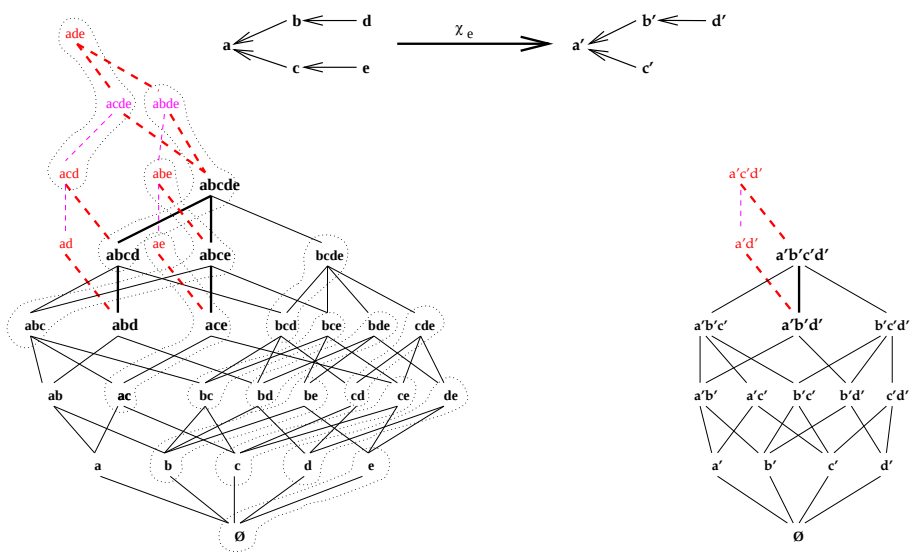


Figure 9: The single point deletion, χ_e applied to the closure space of Figure 4

$\{a'c'd'\}$ because $c \in \{ad\}.\Delta_e$. Observe the role of dependency sets; they insure that this deletion is continuous. For example, $\{abe\}$ maps onto $\{a'b'c'\}$ which is a closed set in $(\mathbf{U}', \varphi'_C)$. And while $\{abe\}$ is not closed in $(\mathbf{U}, \varphi_C)$ its closure $\{abce\} = \{abe\}.\varphi_C$ also maps onto $\{a'b'c'\}$.

Lemma 3.8 *Single point deletions,* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$*, are continuous. I.e. if* $Y.\chi_p$ *is closed, then* $Y.\varphi.\chi_p = Y.\chi_p$.

Proof: Let Y' be closed in $(\mathbf{U}', \varphi')$ and let $Y.\chi_p = Y'$. We assume Y is not closed, else the result follows trivially.
If $p \notin Y$ then $Y.\chi_p = Y = Y'$, so by Lemma 1.8(b), Y is closed wrt. φ.
Let $p \in Y$, so $Y.\chi_p = Y \cup (Y - \{p\}).\Delta_p - \{p\} = Y'$ which is closed wrt. φ'. Consequently by Lemma 1.8(b), $Y \cup (Y - \{p\}).\Delta_p - \{p\}$ must be closed wrt. φ and by Theorem 1.2(a), $Y \cup (Y - \{p\}).\Delta_p$ must be closed wrt. φ as well. Because $Y.\varphi \cap [Y \cup (Y - \{p\}).\Delta_p]$ is closed, $Y.\varphi \subseteq Y \cup (Y - \{p\}).\Delta_p$.

Moreover, $Y.\varphi = Y \cup (Y - \{p\}).\Delta_p$ because $\forall q \in Y \cup (Y - \{p\}).\Delta_p$ either $q \in Y$ or $q \in (Y - \{p\}).\Delta_p$ implying $q \in Y.\varphi$ (because $p \in Y$). Thus, $Y.\varphi.\chi_p = Y \cup (Y - \{p\}).\Delta_p - \{p\} = Y'$ □

Lemma 3.9 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion, then*

$$Y.\varphi.\chi_p \subseteq Y.\chi_p.\varphi'$$

Proof: Since χ_p maps $(\mathbf{U}, \varphi)$ onto $(\mathbf{U}', \varphi')$, where $\mathbf{U}' = \mathbf{U} - \{p\}$ and $\varphi' = \varphi|_{U-\{p\}}$, there exists $Y_0 \subseteq \mathbf{U}$ such that $Y_0 = Y.\chi_p.\varphi'$, and since χ_p is continuous, we may assume Y_0 is closed. Consequently, $Y_0.\chi_p = Y_0 - \{p\} = Y.\chi_p.\varphi'$. Thus, $Y.\chi_p \subseteq Y_0 - \{p\}$ implying $Y \subseteq Y_0$ which is closed. So, $Y.\varphi \subseteq Y_0$ and $Y.\varphi.\chi_p \subseteq Y_0.\chi_p = Y.\chi_p.\varphi'$. □

Theorem 3.10 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion, then*

$$Y.\varphi.\chi_p = Y.\chi_p.\varphi'$$

Proof: (We observe that $Y.\chi_p.\varphi' \subseteq Y.\varphi.\chi_p$ by Corollary 3.7, even though we will not use this in the proof.) If Y is closed, then by Lemma 3.6 $Y.\varphi.\chi_p$ is closed, so $Y.\chi_p.\varphi' = Y.\varphi.\chi_p$. Consequently, we assume Y is not closed in $(\mathbf{U}, \varphi)$. Either $p \in Y.\beta$ or not.
If $p \notin Y.\varphi$ then $Y.\varphi.\chi_p = Y.\varphi = Y.\varphi|_{U-\{p\}} = Y.\chi_p.\varphi'$, so we assume $p \in Y.\varphi$.
If $p \notin Y.\beta$, then $Y.\varphi$ is covered by $Y.\varphi - \{p\}$ and either $p \notin Y$ or $Y - \{p\}$ covers Y. In either case, $Y.\varphi.\chi_p = Y.\varphi - \{p\}$ which by Lemma 3.6 must be closed in $(\mathbf{U} - \{p\}, \varphi|_{U-\{p\}})$. Hence $Y.\varphi.\chi_p = Y.\varphi - \{p\} = Y.\varphi|_{U-\{p\}} = Y.\chi_p.\varphi'$.
Finally, we assume $p \in Y.\beta$ By Lemma 1.4(a), $Y.\varphi$ covers $Y.\varphi - \{p\}$ so $Y.\varphi.\chi_p = Y.\varphi - \{p\}$ which is closed in $(\mathbf{U} - \{p\}, \varphi|_{U-\{p\}})$ as before. But Y need not cover $Y - \{p\}$. If Y does cover $Y - \{p\}$ then $Y.\chi_p = Y - \{p\} \in [Y.\varphi - \{p\}, (Y.\varphi - \{p\}).\beta]$. So, $Y.\varphi.\chi_p = Y.\varphi - \{p\} = (Y - \{p\}).\varphi'$.
If y does not cover $Y - \{p\}$, then $Y.\chi_p = Y \cup (Y - \{p\}).\Delta_p - \{p\} = Y \cup \{ q \mid q \notin (Y.\varphi - \{p\}).\varphi,$ but $q \in Y.\varphi \} - \{p\}$. This latter set is covered by $Y \cup \{ q \mid q \notin (Y.\varphi - \{p\}).\varphi,$ but $q \in Y.\varphi \} \subseteq Y.\varphi$. Hence, $Y.\chi_p \in [(Y - \{p\}).\beta, Y.\varphi - \{p\}]$ and so $Y.\chi_p.\varphi' = Y.\varphi - \{p\} = Y.\varphi.\chi_p$. □

Finally, χ_a takes the connected graph G of Figure 10 onto a disconnected graph G' whose closure lattice is the Boolean algebra on 4 elements. All sets are closed. It is an exact copy of the Boolean algebra below $\{bcde\}$ in $\mathcal{L}_{(U,\varphi_C)}$.

3.4 Lattice Morphisms

The preceding examples clearly indicate that deletions not only change the structure of the graph, they also induce morphisms from one closure lattice to another. In fact, used the lattice morphisms to help illustrate the behavior of the transformation. Consequently, we extend the breadth of our investigations slightly. Let $\mathbf{U} \xrightarrow{f} \mathbf{U}'$ be a transformation of $2^{\mathbf{U}}$

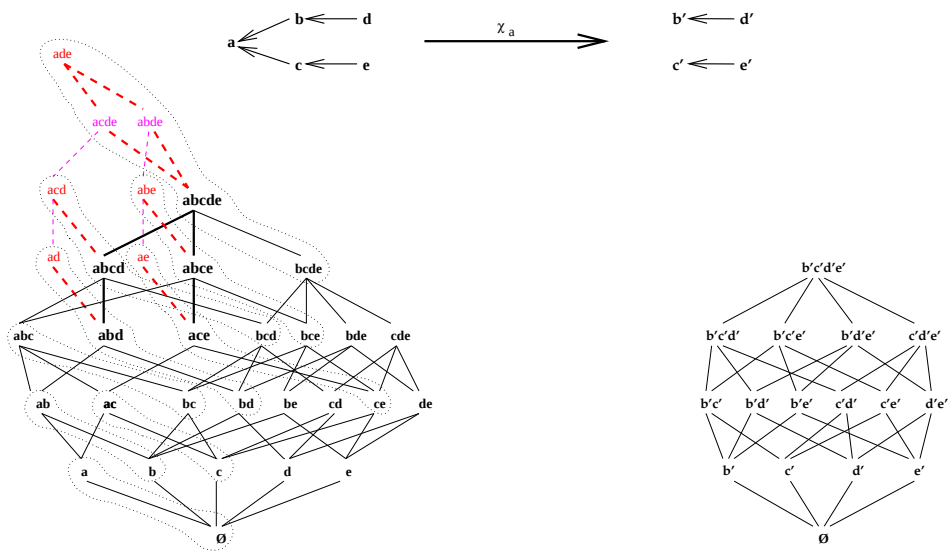


Figure 10: The single point deletion, χ_a applied to the closure space of Figure 4

into $2^{\mathbf{U}'}$. In Section 2, we explored the behavior of f when viewed as operating on the antimatroid closure space $(\mathbf{U}, \varphi)$, as illustrated in Figure 2. Now we formally consider the behavior of transformations on the closure lattice, $\mathcal{L}_{(U,\varphi)}$ as illustrated in Figure 11.

$$\begin{tikzcd}
\mathbf{U} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathbf{U}' \arrow[d, "\varphi'"] \\
(\mathbf{U}, \varphi) \arrow[r, "f"] \arrow[d, "\leq_\varphi"] & (\mathbf{U}', \varphi') \arrow[d, "\leq_{\varphi'}"] \\
\mathcal{L}_{(U,\varphi)} \arrow[r, "f"] & \mathcal{L}_{(U',\varphi')}
\end{tikzcd}$$

Figure 11: Extending f to a lattice morphism

We have seen that all three deletions are closed and continuous. Apparently, all three deletions induce natural lattice morphisms $\mathcal{L}_{(U,\varphi)} \xrightarrow{\chi} \mathcal{L}_{(U',\varphi')}$. Now, as a prelude to demonstrating the lattice morphism properties, we show that all three deletions are order preserving.

Theorem 3.11 *Single point deletions* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *are order preserving, that is if* $X \leq_\varphi Z$ *then* $X.\chi_p \leq_\varphi Z.\chi_p$.

Proof: Let χ denote χ_p. $X \leq_\varphi Z$ implies that $Z \cap X.\varphi \subseteq X \subseteq Z.\varphi$. We must show that $X.\chi \leq_\varphi Z.\chi$, or $Z.\chi \cap X.\chi.\varphi' \subseteq X.\chi \subseteq Z.\chi.\varphi'$ or since χ and φ commute (Theorem 3.10), $Z.\chi \cap X.\varphi.\chi \subseteq X.\chi \subseteq Z.\varphi.\chi$. (Notice the importance of Theorem 3.10, which allows us to draw the argument back into $(\mathbf{U}, \varphi)$.) By Lemma 3.5(b), $X \subseteq Z.\varphi$ implies $X.\chi \subseteq Z.\varphi.\chi$. So, we need only prove the first containment, $Z.\chi \cap X.\varphi.\chi \subseteq X.\chi$, as required by (1).

If Z is closed, $X \leq_\varphi Z$ implies X is closed. And, if X is closed, $X.\varphi.\chi = X.\chi$, whence the first containment is trivial.

We assume neither X nor Z is closed. $X \leq_\varphi Z$ implies by Theorem 1.5 that there exists $Z_X \in [Z.\varphi, Z.\beta]$ such that $Z_X \leq_\varphi Z$ and $Z_X = X \cup (Z.\varphi - X.\varphi)$. Consequently, $Z \subseteq Z_X$, possibly, $Z_X = Z$. *Wolg.* we assume that $Z.\varphi$ covers $X.\varphi$ and $X \prec_\varphi Z_X$, since otherwise we can make a stepwise argument, using the transitivity of $\leq_\varphi$.
We have three cases: (a) $p \notin Z.\varphi$, (b) $p \in Z.\varphi$ but $p \notin X.\varphi$, and (c) $p \in X.\varphi$.

(a) If $p \notin X.\varphi$ then $p \notin Z.\varphi$ so $Z.\chi \cap X.\varphi.\chi = Z \cap X.\varphi \subseteq X = X.\chi$.

(b) Let $p \in X.\varphi, p \notin X.\varphi$. Since $X.\varphi \prec_\varphi Z.\varphi$, $X.\varphi = Z.\varphi - \{p\}$ and $X = Z_X - \{p\}$. Now, $Z \subseteq Z_X$ implies by Lemma 3.5(c) that $Z.\chi \subseteq Z_X.\chi = X.\chi$. So, $Z.\chi \cap X.\varphi.\chi \subseteq X.\varphi$.

(c) $p \in X.\varphi$. Either $X.\varphi - \{p\}$ covers X or X covers $X.\varphi - \{p\}$. In either case, $X.\varphi.\chi = X.\varphi - \{p\}$.
If $p \notin X.\beta$, $Z - \{p\}$ covers Z, so $Z.\chi \cap X.\varphi = Z - \{p\} \cap X.\varphi - \{p\} \subseteq X - \{p\} = X.\chi$.
Let $p \in Z.\beta$ and let $X.\varphi = Z.\varphi - \{q\}, q \neq p$. $Z_X.\chi = Z_X \cup (Z_X - \{p\}).\Delta_p - \{p\}$. Now, $X \subseteq Z_X \cup (Z_X - \{q\}).\Delta_p$, so $p \in (Z_X - \{q\}).\Delta_q$ implying $q \notin (Z_X - \{p\}).\Delta_p$. Thus, $Z.\chi \cap X.\varphi.\chi \subseteq Z_X.\chi \cap X - \{p\} \subseteq X.\chi$. $\square$

A deletion χ_p maps a closure space $(\mathbf{U}, \varphi)$ onto a subset of itself $(\mathbf{U}', \varphi|_{U-\{p\}})$, hence it maps the lattice $\mathcal{L}_{(U,\varphi)}$ onto the lattice $\mathcal{L}_{(U',\varphi')}$. We are interested in the nature of these lattice morphisms. First we establish that

Lemma 3.12 *In* $\mathcal{L}_{(U,\varphi)}$

(a) $X \cap Z \subseteq X \wedge Z \subseteq X.\varphi \cap Z.\varphi$, *with equality if* X, *or* Z, *is closed.*

(b) $(X \wedge Z).\varphi = X.\varphi \wedge Z.\varphi = X.\varphi \cap Z.\varphi$.

Proof:

(a) $X \wedge Z \leq_\varphi X$ implies $(X \wedge Z).\varphi \cap X \leq_\varphi X \wedge Z \leq_\varphi X.\varphi$
$X \wedge Z \leq_\varphi Z$ implies $(X \wedge Z).\varphi \cap Z \leq_\varphi X \wedge Z \leq_\varphi Z.\varphi$
The last two containments together imply $X \wedge Z \subseteq X.\varphi \cap Z.\varphi$, while the first two imply $(X \wedge Z).\varphi \cap X \cap Z \subseteq X \wedge Z$. or since $X \wedge Z \subseteq (X \wedge Z).\varphi$, $X \cap Z \subseteq X \wedge Z$.
If X is closed, $X \wedge Z \leq_\varphi X$ implies $X \wedge Z$ is closed and $X \wedge Z \subseteq X$. Consequently, $(X \wedge Z).\varphi \cap X = (X \wedge Z) \cap Z = X \wedge Z$.

(b) That $X.\varphi \wedge Z.\varphi = X.\varphi \cap Z.\varphi$ follows from the definition of closure. If either X or Z is closed, the result follows directly from (a). Assume neither X nor Z is closed, and that $X.\varphi \neq Z.\varphi$ (else the conclusion is trivial).
Since $X \wedge Z \subseteq X.\varphi \cap Z.\varphi$, $(X \wedge Z).\varphi \subseteq X.\varphi \cap Z.\varphi$, or $(X \wedge Z).\varphi \leq_\varphi X.\varphi \cap Z.\varphi$. But, $X.\varphi \leq_\varphi X$ and $Z.\varphi \leq_\varphi Z$ imply that $X.\varphi \wedge Z.\varphi \leq_\varphi X \wedge Z$, and by Theorem 1.5, $X.\varphi \wedge Z.\varphi \leq_\varphi (X \wedge Z).\varphi$. So $X.\varphi \wedge Z.\varphi = (X \wedge Z).\varphi$. $\square$

Recall that a lattice morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is called a **homomorphism** if it preserves both the meet ($\wedge$) and join ($\vee$) operators; it is called a **meet homomorphisms** if it just preserves the meet operator, that is $(x \wedge z).f = x.f \wedge z.f$, for all $x, z \in \mathcal{L}$. Our goal now is to show that deletions induce a morphism on the closure lattices that is midway between the two. It is a meet homomorphism that preserves the join operator under specific conditions. We first establishe that every deletion is a meet homomorphism.

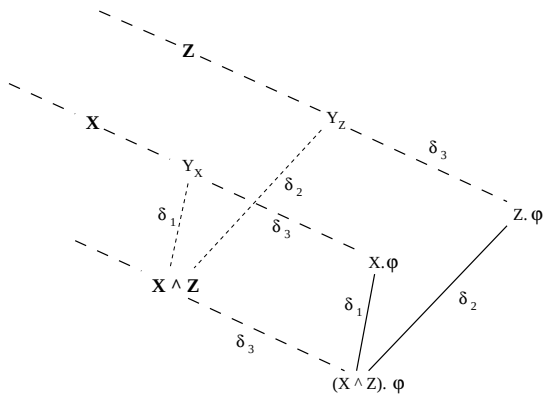
Theorem 3.13 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion, then*

$$(X \wedge Z).\chi_p = X.\chi_p \wedge Z.\chi_p$$

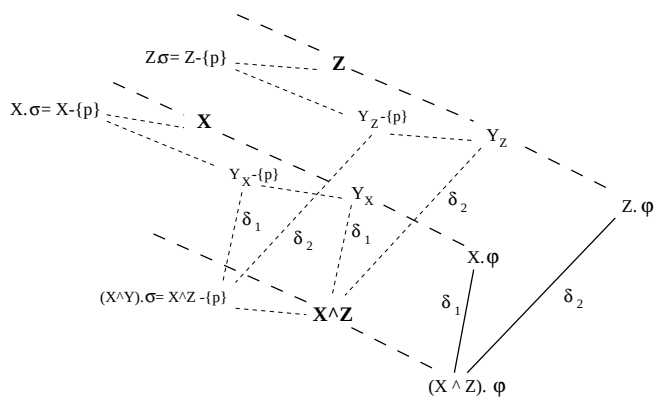
Proof: Let χ denote χ_p. Let X, or Z, be closed. Then $X \wedge Z$ is closed and $X \wedge Z = X \wedge Z.\varphi = X \cap Z.\varphi$ by Lemma 3.12(a). Consequently, by Lemma 3.5(a) and Theorem 3.10

$$\begin{aligned}
(X \wedge Z).\chi &= (X \cap Z.\varphi).\chi = X.\chi \cap Z.\varphi.\chi \\
&= X.\chi \cap Z.\chi.\varphi' = X.\chi \wedge Z.\chi.\varphi' \\
&= X.\chi \wedge Z.\chi
\end{aligned}$$

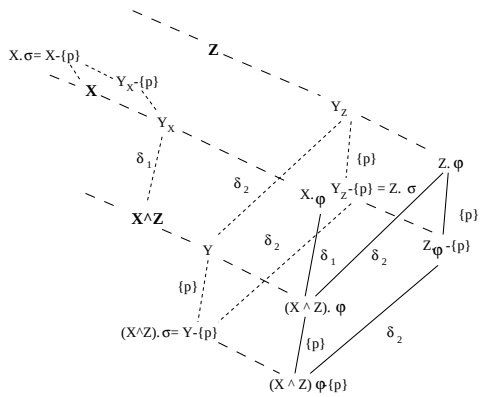
So we assume neither X nor Z is closed.
If $X \wedge Z$ is closed, then $X \wedge Z = X.\varphi \wedge Z.\varphi$, and a slight extension of the argument above will suffice. We assume $X \wedge Z$ is not closed and that $X.\varphi \neq Z.\varphi$, since otherwise X, Z are in the same interval $[X.\varphi, X.\beta]$, which is a Boolean algebra and the conclusion is evident.
Since $X \wedge Z$ is not closed it is in an interval $[(X \wedge Z).\varphi, (X \wedge Z).\beta]$. By Lemma 3.12(b), $(X \wedge Z).\varphi = X.\varphi \wedge Z.\varphi$, and by Theorem 1.5, there exists unique elements $Y_X \in [X.\varphi, X.\beta], Y_Z \in [Z.\varphi, Z.\beta]$ such that $X \wedge Z \leq_\varphi Y_X \leq_\varphi X$ and $X \wedge Z \leq_\varphi Y_Z \leq_\varphi Z$. So, $X \wedge Z = Y_X \wedge Y_Z$ as shown in the figure. It suffices to show that $(X \wedge Z).\chi = (Y_X \wedge Y_Z).\chi = Y_X.\chi \wedge Y_Z.\chi$, and it is easiest to follow the structure of the proof if we assume that $X.\varphi$ and $Z.\varphi$ both cover $(X \wedge Z).\varphi$ *I.e.* $|\delta_1| = |\delta_2| = 1$. No generality is lost thereby.
We first suppose that $p \notin (X \wedge Z).\varphi$, so $(X \wedge Z).\chi = X \wedge Z$. If $p \notin X.\varphi$ then $X.\chi = X$ and if $p \notin Z.\varphi$ then $Z.\chi = Z$. Clearly, if p is an element of neither, then the conclusion follows trivially. Suppose $p \in X.\varphi$, implying $\delta_1 = \{p\}$ and $Y_X.\chi = X \wedge Z$. $\delta_1 \neq \delta_2$ else $X.\varphi = Z.\varphi$. So $p \notin Y_Z$, $Y_Z.\chi = Y_Z$, and $(Y_X \wedge Y_Z).\chi = (X \wedge Z).\chi = X \wedge Z = X \wedge Z \wedge Y_Z = Y_X.\chi \wedge Y_Z.\chi$.



Next, we assume $p \in (X \wedge Z).\varphi$. There are two cases. Either $p \in (X \wedge Z).\beta$, or not. First, we suppose $p \notin (X \wedge Z).\varphi$, so neither δ_1 nor $\delta_2 = \{p\}$. $(X \wedge Z).\chi = (X \wedge Z) - \{p\} \in [(X \wedge Z).\varphi, (X \wedge Z).\beta]$. By



Theorem 1.5, $(X \wedge Z) - \{p\} \cup \{\delta_1\} \in [X.\varphi, X.\beta]$ and covers $(X \wedge Z) \cup \{p\} = Y_X$. $(X \wedge Z) - \{p\} \cup \{\delta_2\} \in [Z.\varphi, Z.\beta]$ and covers $(X \wedge Z) \cup \{p\} = Y_Z$. Consequently, $Y_X.\chi \wedge Y_Z.\chi = ((X \wedge Z) \cup \{\delta_1\} - \{p\}) \wedge ((X \wedge Z) \cup \{\delta_2\} - \{p\}) = (X \wedge Z) - \{p\} = (X \wedge Z).\chi$.
This leaves just the case where $p \in (X \wedge Z).\beta$. Let $p \in (X \wedge Z).\beta$. Then $(X \wedge Z).\varphi$ covers $(X \wedge Z - \{p\}).\varphi$. If $X \wedge Z$ covers $Y \wedge Z - \{p\} \in [(X \wedge Z - \{p\}).\varphi, (X \wedge Z - \{p\}).\beta]$, then we can use Theorem 1.5 as before to find the unique minimal Y_X and Y_Z such that $X \wedge Z - \{p\} \leq_\varphi Y_X \leq_\varphi X$ and $X \wedge Z - \{p\} \leq_\varphi Y_Z \leq_\varphi Z$ with $(X \wedge Z).\chi = X \wedge Z - \{p\} = Y_X \wedge Y_Z = X.\chi \wedge Z.\chi$.
But, if $p \in (X \wedge Z).\beta$, $X \wedge Z$ need not cover $X \wedge Z - \{p\}$. However, $p \in (X \wedge Z).\beta$ implies $(X \wedge Z).\varphi$ covers $(X \wedge Z - \{p\}).\varphi$, and by Lemma 3.2, $X \wedge Z \cup (X \wedge Z - \{p\}).\Delta_p$ must cover $X \wedge Z \cup (X \wedge Z - \{p\}).\Delta_p - \{p\} = (X \wedge Z).\chi$. Let $Y = X \wedge Z \cup (X \wedge Z - \{p\}).\Delta_p \in [(X \wedge Z).\varphi, (X \wedge Z).\beta]$, so $Y - \{p\} = X \wedge Z \cup (X \wedge Z - \{p\}).\Delta_p - \{p\} = (X \wedge Z).\chi$ as in the figure. We have more cases; either $p \in X.\beta$, $p \in Z.\beta$, or not. In the figure we show $p \notin X.\beta$ and $p \in Z.\beta$, but any of the four combinations are possible. We assume this configuration. As before, there exist unique minimal $Y_X \in X.\varphi$ and



$Y_Z \in Z.\varphi$ such that $X \wedge Z \ \leq_\varphi \ Y_X \ \leq_\varphi \ X$ and $Y \ \leq_\varphi \ Y_Z \ \leq_\varphi \ Z$.

$$\begin{aligned}(X \wedge Z).\chi = Y - \{p\} &= Y \wedge Y_Z - \{p\} \\ &= Y_X - \{p\} \ \wedge \ Y_Z - \{p\} \\ &= X - \{p\} \ \wedge \ Y_Z - \{p\} = X.\chi \wedge Z.\chi\end{aligned}$$

The other three cases are handled similarly. □

Theorem 3.11 could have be stated as a corollary to this lemma; or alternatively, we could have used that theorem to first demonstrate that $(X \wedge Z).\chi_p \ \leq_{\varphi'} \ X.\chi_p \wedge Z.\chi_p$. We thought that two different proofs, one which made abundant use of Theorem 3.10 and one which made no mention of it whatever, were valuable.

Some of the subtleties found in the last few cases of Theorem 3.13 are illustrated by Figure 12. Observe that $\{df\} = \{bdf\} \wedge \{cdf\}$ so $d \in \{df\}.\beta$ and $\{df\}.\chi_d = \{f\}.\Delta_d - \{d\} = \{e'f'\}$. Now $\{df\}$ does not cover $\{df\}.\chi_d$ in $\mathcal{L}_{(U,\varphi)}$ as is so often the case. Moreover, while $\{df\}$ is covered by both $\{bdf\}$ and $\{cdf\}$ in $\mathcal{L}_{(U,\varphi)}$, neither $\{bdf\}.\chi_d = \{b'f'\}$ nor $\{cdf\}.\chi_d = \{c'f'\}$ cover $\{df\}.\chi_d = \{e'f'\}$. Single point deletions need not preserve covering relationships. However,

Lemma 3.14 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion, and*

(a) $X \vee Z$ *is closed,*

(b) $X \vee Z$ *covers* X *and* Z

then

$$(X \vee Z).\chi_p = X.\chi_p \vee Z.\chi_p$$

Proof: Let σ denote χ_p. Because σ is order preserving, $X.\sigma \vee Z.\sigma \ \leq_{\varphi'} \ (X \vee Z).\sigma$. And because, the sublattice of closed subsets is lower semimodular, X and Z each cover $X \wedge Z$.

If $p \notin X \vee Z$, then $p \notin X$ or Z, so the conclusion follows trivially.

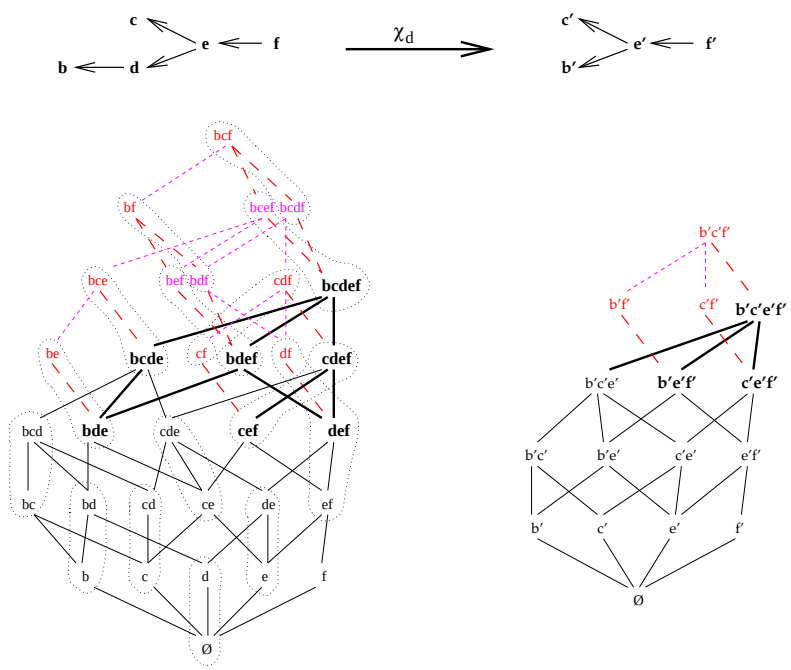
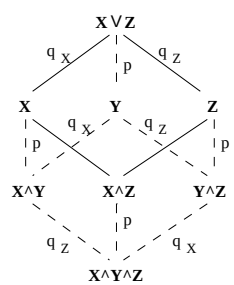


Figure 12: A single point deletion, χ_d

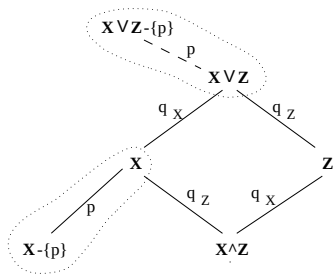
Let $p \in X \vee Z$. Let $X = X \vee Z - \{q_X\}, Z = X \vee Z - \{q_Z\}$. Suppose $q_X = p$ (or $q_Z = p$), then $X.\sigma = X = (X \vee Z).\sigma$. By lower semimodularity, $X \wedge Z = Z - \{p\}$. So, $X.\sigma = X \wedge Z = (X \wedge Z).\sigma$ implying $(X \vee Z).\sigma \leq_{\varphi'} X.\sigma \vee Z.\sigma$.

If neither q_X nor $q_Z = p$, we must consider cases.
If $p \in (X \vee Z).\beta$ then there exists Y such that $Y = X \vee Z - \{p\}$. By meet distributivity (see Edelman [4] or Pfaltz [13]) we have the substructure shown in the figure where the covering edges



are labeled with the appropriate elements. $(X \vee Z).\sigma = Y' = Y.\sigma, X.\sigma = X' \wedge Y' = (X \wedge Y).\sigma$, $Z.\sigma = Y' \wedge Z' = (Y \wedge Z).\sigma$ and $(X \wedge Z).\sigma = X' \wedge Y' \wedge Z' = (X \wedge Y \wedge Z).\sigma$. The conclusion follows.

Finally we suppose $p \notin (X \vee Z).\beta$, implying $X \vee Z - \{p\} \in [(X \vee Z).\varphi, (X \vee Z).\beta]$ and $(X \vee Z).\sigma = (X \vee Z - \{p\}).\sigma$.
If $p \notin X.\beta$ (or $p \notin Z.\beta$) there is no problem since $X.\sigma = (X - \{p\}.\sigma$, where $X - \{p\} \in [X.\varphi, X.\beta]$ and by Theorem 1.5 $X - \{p\} \prec_\varphi X \vee Z - \{p\}$.
If $p \in X.\beta$ (or $p \in Z.\beta$) we have the configuration shown below. Here $X.\sigma = (X - \{p\}).\sigma =$



$X \vee - \{q_X\} - \{p\} = (X \vee Z).\sigma - \{q'_X\}$. By Theorem 1.2(a), this is a cover in $(\mathbf{U}', \varphi')$ because

$$\begin{aligned}(X.\sigma \cup \{q_X\}).\varphi' &= X.\sigma.\varphi' \cup \{q_X\} \\ &= X.\varphi.\sigma \cup \{q_X\}. \qquad \square\end{aligned}$$

That this result cannot be generalized to all of $\mathcal{L}_{(U,\varphi)}$ is also demonstrated by Figure 12 where $\{cdf\}$ which is not closed covers $\{df\}$ and $\{cdef\}$; but $\{cdf\}.\chi_d = \{c'f'\}$ does not cover $\{df\}.\chi_d = \{e'f'\}$.

A lattice morphism $\mathcal{L} \xrightarrow{f} \mathcal{L}'$ is said to be a **lower semihomomorphism**, or LSH, if for all $x, z \in \mathcal{L}$, $(x \wedge z).f = x.f \wedge z.f$ and $x \prec x \vee z$, $z \prec x \vee z$ together imply $(x \vee z).f = x.f \vee z.f$. Properties of lower semihomomorphisms can be found in [12].[7]

Corollary 3.15 *If* $(\mathbf{U}, \varphi) \xrightarrow{\chi_p} (\mathbf{U}', \varphi')$ *is a single point deletion then*

$$\mathcal{L}_{(U,\varphi)} \xrightarrow{\chi_p} \mathcal{L}_{(U',\varphi')}$$

is a lower semihomomorphism when restricted to the sublattice of closed subsets.

Proof: This is an immediate consequence of Theorem 3.13 and Lemma 3.14. $\square$

Although deletions, χ_p, are lower semihomomorphic only on the sublattice of closed subsets, it is sufficient, because antimatroid closure spaces are completely determined by just this sublattice. (The generators, $Y.\beta$, of any closed subset Y can be determined by examining the closed sets covered by Y.) We therefore say that deletions χ induce a lower semihomomorphic transformation of one closure space onto another.

[7] We caution the reader that [12] assumes that all lattices, which are called G-lattices, are atomic, that is, all singleton elements are closed. This is a characteristic of all convex geometries, but need not be true in general. For example, the ideal path closures φ_L and φ_R yield atomic closure spaces only if they are totally disconnected.

4 Observations

One can make a number of observations based on the preceding sections. If, as is the case with graph homomorphisms, one defines a transformation of discrete spaces in terms of a point map f on the set $\mathbf{U}$ of elements, then one can't have a deletion χ_p in which a point $p \in \mathbf{U}$ simply "disappears". It would not be well-defined over all of $\mathbf{U}$. But, because χ is defined on the power set $2^{\mathbf{U}}$, one can have $\{p\}.\chi_p = \emptyset$. Now, deletions and their induced lattice morphisms seem natural.

We have considered only single point deletions simply because they are so much clearer, and because they are basic to the deletion concept. But, single point deletions can be composed. Figure 13 illustrates the composition of χ_c with χ_d. Note the sublattice $[\emptyset, c, d, cd]$

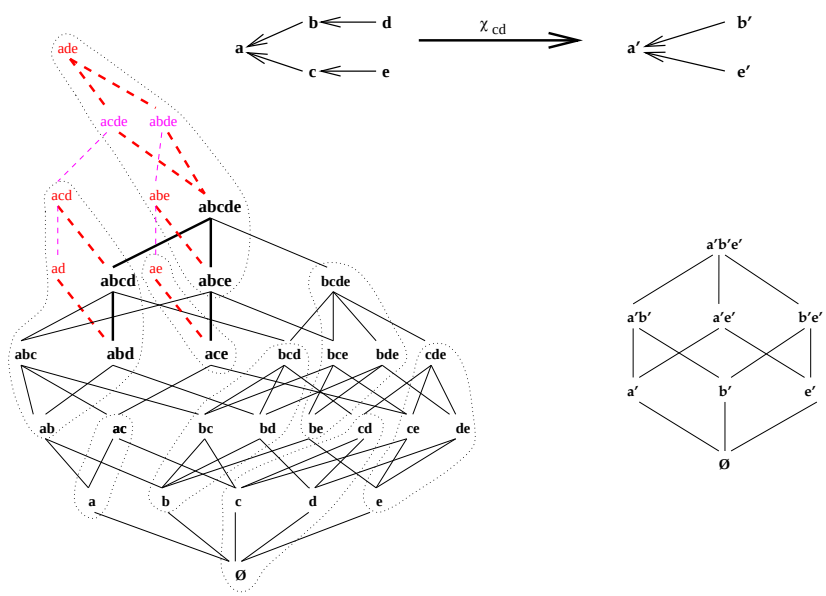


Figure 13: The composition of χ_c and χ_d applied to the closure space of Figure 4

of $(\mathbf{U}, \varphi)$ which maps onto $\emptyset$ of $(\mathbf{U}', \varphi')$. Following [12], we call this the **kernel** of the deletion χ. If X and Z are in the kernel of a deletion, then because χ is a semimodular homomorphism, $X \wedge Z$ must be in the kernel, and $X \vee Z$ is in the kernel if and only if it covers both X and Z. We chose to illustrate χ_{cd} because it is clean and well-behaved. The reader is encouraged to visualize χ_{ae}, the composition of χ_a with χ_e. It is far more interesting.

Based on the deletion concept, one can now define families of antimatroid closure spaces. An early, but inadequate, attempt to do this in terms of "convex" graph homomorphisms can be found in [14]. In essence, except for some simple, common closure spaces such as n

independent, unrelated points, or n totally ordered points, the lattice structure of different closure operators appear to be quite distinct. For example, let $\mathbf{U}$ consist of n points which have an arbitrary partial order $\leq$ on them. In general, the lattices of $(\mathbf{U}, \varphi_L)$ and $(\mathbf{U}, \varphi_R)$ are quite distinct from $(\mathbf{U}, \varphi_C)$ as well as each other. These, in turn, are structurally different from the lattice obtained by considering the convex geometry of the same points projected onto a plane, whose structure must be different from the convex geometry if the points were projected into 3-space. (In these we implicitly assumed a Euclidean geometry; other geometries may yield other antimatroid closure spaces.)

These families of closure lattices are quite interesting. Members within one family have similarities which tend to distinguish them from members of another family. For example, if we call a generating set $X.\beta$ **non-trivial** when $X.\beta \subset X.\varphi$, then if $X.\beta$ is non-trivial in a planar convex geometry, $|X.\beta| \geq 3$; while $X.\beta$ non-trivial in a 3-dimensional convex geometry ensures that $|X.\beta| \geq 4$. This characteristic difference is rather obvious; others seem to be more subtle. We conjecture that, except for the kinds of special cases mentioned above, there exist no semimodular lattice homomorphisms from closure lattices of one family of antimatroid closure spaces onto the closure lattices of another family.

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