Closure Spaces in the Plane*

John L. Pfaltz

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Abstract

When closure operators are defined over figures in the plane, they are normally defined with respect to convex closure in the Euclidean plane. This report concentrates on discrete closure operators defined over the discrete, rectilinear plane.

Basic to geometric convexity is the concept of a geodesic, or shortest path. Such geodesics can be regarded as the closure of two points. But, given the usual definition of geodesic in the discrete plane, they are not unique. And therefore convex closure is not uniquely generated.

We offer a different definition of geodesic that is uniquely generated, although not symmetric. It results in a different geometry; for example, two unbounded lines may be parallel to a third, yet still themselves intersect. It is a discrete geometry that appears to be relevant to VLSI design.

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1 Overview

In this paper we examine discrete anti-matroid closure spaces in the plane. These are discrete systems, $\mathbf{U}$, which have a closure operator, φ, that satisfies the usual three closure axioms:[1] for all , $Y \subseteq \mathbf{U}$,

$Y \subseteq Y.\varphi$,
$X \subseteq Y$ implies $X.\varphi \subseteq Y.\varphi$, and
$Y.\varphi.\varphi = Y.\varphi$.

A set Z for which $Z.\varphi = Z$ is said to be **closed**. A minimal subset $X \subseteq Z$ such that $X.\varphi = Z.\varphi$ is called a **generator**, or a **kernel**[2] and denoted $Z.\kappa$. The anti-matroid properties come from a fourth *anti-exchange* axiom,

$p, q \notin Y.\varphi$ and $q \in (Y \cup \{p\}).\varphi$ imply $p \notin (Y \cup \{q\}).\varphi$.

Many of these properties have been described in [14]. In particular it is shown that in an anti-matroid closure space every set is *uniquely generated.*

If instead the closure system satisfies the *exchange* axiom

$p, q \notin Y.\varphi$ and $q \in (Y \cup \{p\}).\varphi$ imply $p \in (Y \cup \{q\}).\varphi$.

one has a matroid, or generalized vector space. Matroid properties have been widely studied, for example [19] or [20].

We consider closure spaces, rather than more classical topological spaces, because for many computer applications closure is easier to define than open sets. At one time it had been thought tht there were relatively few uniquely generated closure spaces. But, in [12] it is shown that there exist asymptotically at least $O((2^n)^{n/2})$ distinct anti-matroid closure spaces on n points. That is, if $n > 10$ there exist at least $n^n << O((2^n)^{n/2})$ distinct closure operators. The closure concept is very rich.

One of the most familiar closure operators is the *convex hull* operator. The first 3 axioms are easily verified. Convexity may, or may not, satisfy the anti-exchange axiom depending on the definition of geodesics (or shortest paths) in the space. Usually, when geodesic paths are unique, then convex closure is uniquely generated.[3] Here we consider only planar closure spaces because many computer applications, ranging from image processing to VLSI layout, are planar in nature. However, the same ideas can be extended to three, or more, dimensions.

All the systems we will consider are discrete in the sense that the closed sets of interest

[1] Throughout this paper, set-valued operators are expressed using suffix notation.

[2] Many terms are found in the literature for these minimal generating sets depending on ones approach. With convex closure in discrete geometry one speaks of *extreme points* [4], a term we will use in this paper as well. With respect to transitive closures in relational algebras one calls them the *irreducible kernel* [2]. In [14], we called them *generators*, but denoted them with the symbol β, suggestive of *basis* the generators of a matroid space.

[3] One must be careful. Shortest paths in chordal graphs are not unique closure generators; but monophonic paths are [5].

are finitely generated. However, there are two very distinct families of planar closure spaces — those which assume a continuous, Euclidean plane and those which assume a discrete rectilinear grid.

2 Discrete Geometries in the Euclidean Plane

Convexity concepts and convex geometries in n-dimensional Euclidean space, E^n, have been well studied. Although the closed sets of the space itself may not be discrete, it is usually assumed that the number of generators of each closed set is finite. Indeed, this is the key to Dantzig's approach to the solution of linear programming inequalities. There may be an infinite number of solutions, but the linear inequalities define half planes that create a convex polytope representing the space of feasible solutions. Since this polytope has only a finite number of generating vertices, or extreme points, it is only these points in the solution space that need be tested for optimality [3]. The convex subsets of E^n constitute an antimatroid closure space [7], and since the relative restriction of an antimatroid closure operator is again uniquely generated [8, 15], any discrete subset under ordinary Euclidean convexity will be a closure space.

Convexity is a closure concept. The convex hull operator is the closure operator. The anti-matroid properties of convex hull operators has been repeatedly discovered, even though they may be named differently [11]. An excellent theory of convex geometries can be found in [4]. We adopt many of their ideas and some of its terminology. In particular, we treat "extreme point" and "generator" as synonomous when talking about geometric closure.

A set Y is said to be **convex** if every geodesic between two points y_1, y_2 is completely contained within Y. In the Euclidean plane, geodesics are straight lines so the convex, or closed, sets are polygonal. Frequently, we are more interested in the layout of a relatively few elements distributed in the Euclidean plane. Figure 1(a) illustrates a small convex geometry of just 6 points. A major contribution of [14] was to demonstrate that one can partially order *all* the subsets of a closure space; and that its subsets, so ordered, form a regular lattice called its *closure lattice*. Figure 1(b) is the corresponding closure lattice, $\mathcal{L}$. The closed sets form a lower semi-modular sublattice, $[\emptyset, abcdef]$, denoted by the solid edges. There are only 11 non-trivial generating sets. These are connected to their corresponding closures by dashed lines generally slanting from upper left to lower right. Because most configurations are closed, $\mathcal{L}$ is excessively busy and reveals little of the internal structure. With more points, a lattice structure becomes more valuable.

As noted earlier, convex hull operators over points in E^n are well known. Much less well known is convexity defined on lines in E^n; see [6, 1].

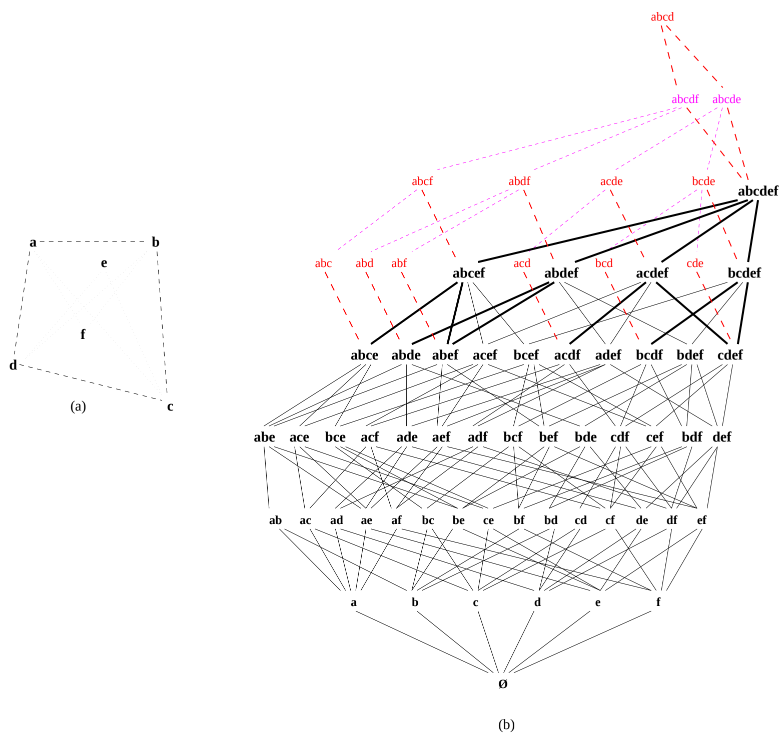


Figure 1: A convex geometry and its closure lattice

3 Discrete Geometries over a Rectilinear Grid

The Euclidean plane is easily represented as a rectilinear grid by restricting the point coordinates, $p_{i,k}$, to be integer. In this case, it is customary to use the city block metric, that is $\delta(p_{i,k}, q_{m,n}) = |i - m| + |k - n|$. Other metrics can be defined [17, 10], but this is the most common. Given a city block metric, most paths between p and q within the rectangle of Figure 2 are geodesic. One can define this rectangle to be the convex hull generated by $\{pq\}$. It is a closure operator, but it is clearly not uniquely generated. The opposite corners also constitute a generating set. To generate reasonable closure and geodesic concepts in a rectilinear grid requires some care. Our approach will be different from that found in [16, 18, 21].

We begin by first defining sixteen direction operators. Relative to any single element y, we first have the four operators:

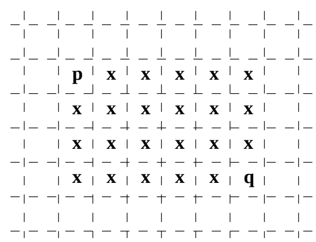


Figure 2: City block closure, $\delta(p, q) = 4 + 6 = 10$

$$\left.\begin{array}{l} y.U \\ y.D \\ y.L \\ y.R \end{array}\right\} \begin{array}{l} \textit{all elements } x \textit{ precisely} \\ \textit{above, below,} \\ \textit{to the left and} \\ \textit{to the right of } y \end{array}$$

These operators denote all elements in the same column or row as y; that is, if y is denoted by $y_{m,n}$ then $y.U = \{x_{i,n} | i < m\}$, $y.D = \{x_{i,n} | i > m\}$, $y.L = \{x_{m,k} | k < n\}$, and $y.R = \{x_{m,k} | k > n\}$. Next we have four more operators:

$$\left.\begin{array}{l} y.UL \\ y.UR \\ y.DL \\ y.DR \end{array}\right\} \begin{array}{l} \textit{all elements } x \textit{ anywhere} \\ \textit{in the upper left,} \\ \textit{upper right, lower left,} \\ \textit{and lower right quadrants} \end{array}$$

Or algebraically, $y.UL = \{x_{i,k} | i < m, k < n\}$, $y.UR = \{x_{i,k} | i < m, k > n\}$, $y.DL = \{x_{i,k} | i > m, k < n\}$, and $y.DR = \{x_{i,k} | i > m, k > n\}$.

These eight operators are called the **distant direction** operators because they denote collections of elements at any distance in the indicated direction. The next eight operators are called **neighborhood direction** operators because they denote precisely the singleton element in the indicated direction. The first four are:

$$\left.\begin{array}{l} y.u \\ y.d \\ y.l \\ y.r \end{array}\right\} \begin{array}{l} \textit{the element } x \textit{ immediately} \\ \textit{above, below,} \\ \textit{to the left and} \\ \textit{to the right of } y \end{array}$$

The last four operators are:

$$\left.\begin{array}{l} y.ul \\ y.ur \\ y.dl \\ y.dr \end{array}\right\} \begin{array}{l} \textit{the element } x \textit{ diagonally} \\ \textit{to the upper left,} \\ \textit{upper right, lower left,} \\ \textit{and lower right of } y \end{array}$$

Again, if one is using an algebraic notation to denote elements, we have $y.u = \{x_{i,n} | i = m-1\}$, $y.d = \{x_{i,n} | i = m+1\}$, $y.l = \{x_{m,k} | k = n-1\}$, $y.r = \{x_{m,k} | k = n+1\}$, $y.ul = \{x_{i,k} | i = m-1, k = n-1\}$, $y.ur = \{x_{i,k} | i = m-1, k = n+1\}$, $y.dl = \{x_{i,k} | i = m+1, k = n-1\}$, and $y.dr = \{x_{i,k} | i = m+1, k = n+1\}$. The 8 elements denoted by the eight neighborhood direction operators are collectively called the **neighborhood** of y, denoted $y.\eta$. These distant and neighborhood direction operators are graphically illustrated in Figure 3.

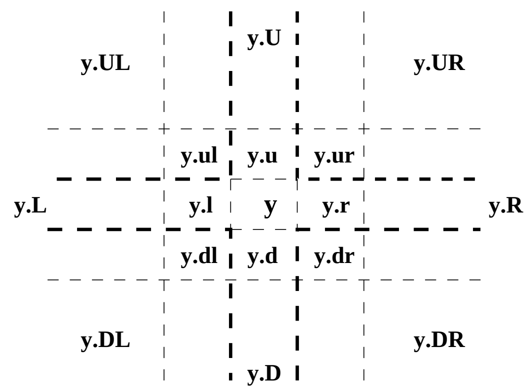


Figure 3: The direction operators

A closure operator is usually defined by assuming that one or more elements belong to the closure, then describing by implication which additional elements must also belong to the closure. First, we define eight directional **closure implications**. Let $y, z \in Y.\varphi$ then

$$\begin{array}{ll} (a) & z \in y.U \text{ implies } y.u \in Y.\varphi \\ (b) & z \in y.D \text{ implies } y.d \in Y.\varphi \\ (c) & z \in y.L \text{ implies } y.l \in Y.\varphi \\ (d) & z \in y.R \text{ implies } y.r \in Y.\varphi \\ (e) & z \in y.UL \text{ implies } y.ul \in Y.\varphi \\ (f) & z \in y.UR \text{ implies } y.ur \in Y.\varphi \\ (g) & z \in y.DL \text{ implies } y.dl \in Y.\varphi \\ (h) & z \in y.DR \text{ implies } y.dr \in Y.\varphi. \end{array} \quad (1)$$

One can define a closure operator, φ by asserting that all eight implications hold. It will satisfy the 3 basic closure axioms, but it won't be uniquely generated; it won't be an antimatroid. To see this, assume that closure φ is defined by all of the implications (a) - (d) and (e) - (h), then the configuration of Figure 4 is convex. Generating elements are denoted by e, corresponding to the *extreme* point notation introduced in the preceding section. We have additionally circled them for visual emphasis. However, there are two distinct minimal generating sets, as indicated by (a) and (b). The reader should convince himself that both

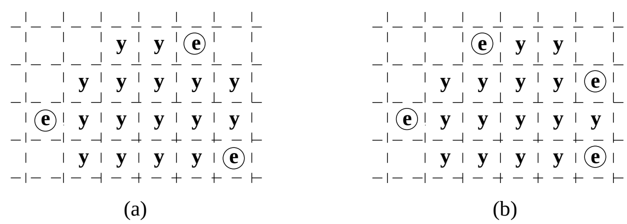


Figure 4: Two generating sets for the same "convex" configuration

sets generate the same closure, and that in fact only implications (e) and (h) cause trouble because they correspond to reciprocal directions.

However, a closure operator, φ, for which implications (a)-(d) and any *two* of implications (e)-(f) (provided they are not reciprocal directions) will constitute a uniquely generated, anti-matroid closure operator. We designate the four resulting closure operators by $\varphi_{U(LR)}$, $\varphi_{(UD)R}$, $\varphi_{D(LR)}$, and $\varphi_{(UD)L}$, where the subscript $U(LR)$ indicates the implications (e) and (f) are the only "diagonal" closures and other subscripts are similar.[4] For examples in the remainder of this paper we will be using $\varphi_{U(LR)}$, unless some other planar closure is clearly indicated. Observe that Figure 4(a) is closed only with respect to $\varphi_{(UD)L}$. The closure set generated by the 4 extreme elements of Figure 4(b) cannot be obtained with respect to any of the four uniquely generated closures.

Theorem 3.1 *The closure operator $\varphi_{U(LR)}$, consisting of the four orthogonal directional implications and two non-reciprocal diagonal directional implications is uniquely generated.*

Proof: Specifying $\varphi_{U(LR)}$ in no way limits the generality of this result. The proof with respect to any of the other non-reciprocal closure operators would be similar.
Let $Y.\varphi$ be any closed set such that $p, q \notin Y.\varphi$. Assume that $q \in (Y \cup \{p\}).\varphi$, implying that q is on a geodesic path between some $y \in Y.\varphi$ and p. Suppose q (and y) $\in p.U$ (or $p.D$ or $p.L$ or $p.R$) then $y \in q.U$, but $p \in q.D$. If $p \in (Y \cup \{q\}).\varphi$ there must exist some $y' \in p.D$ such that p is on the geodesic between q and y'. But now, p and q are on a geodesic between y and y' contradicting the assertion that $p, q \notin Y.\varphi$.
So we may assume that q is in some directional quadrant with respect to p, say $q \in p.UL$. Further, there exists some $y \in q.UL \subseteq p.UL$ such that q is on the geodesic between p and y. Now $p \in q.DR$. $\varphi_{U(LR)}$ is a non-reciprocal closure operator, so if there exists a $y' \in Y.\varphi$ such that p lies on a geodesic between y' and q, we must have p (and q) in $y'.UL$. But now, p and q lie on a geodesic between y and y', again contradicting the assumption the $p, q \notin Y.\varphi$.

[4]This is somewhat similar to the case of acyclic graphs, in which there are at least two closure operators, φ_L and φ_R [13, 14].

Consequently, $p \notin (Y \cup \{q\}).\varphi$; $\varphi_{U(LR)}$ is uniquely generated. □

Re-examining Figure 4 can be worthwhile. In the upper right hand corner of Figure 4(a) we have $y \in e.dl$ and in Figure 4(b) $y \in e.ur$. Readily, these two configurations can occur only if φ is a reciprocal closure operator. The assertion that the 4 indicated extreme elements of Figure 4(b) could not possibly have generated that set with respect to either of $\varphi_{U(LR)}, \varphi_{D(LR)}, \varphi_{(UD)L}$ or $\varphi_{(UD)R}$ needs only an examination of the upper left and lower right portions where the same reciprocal inclusion also occurs.

Figure 5 illustrates the variety of closed sets that can be generated by the three extreme elements of Figure 4(a). Readily, uniquely generated, convex closure in a rectilinear grid is

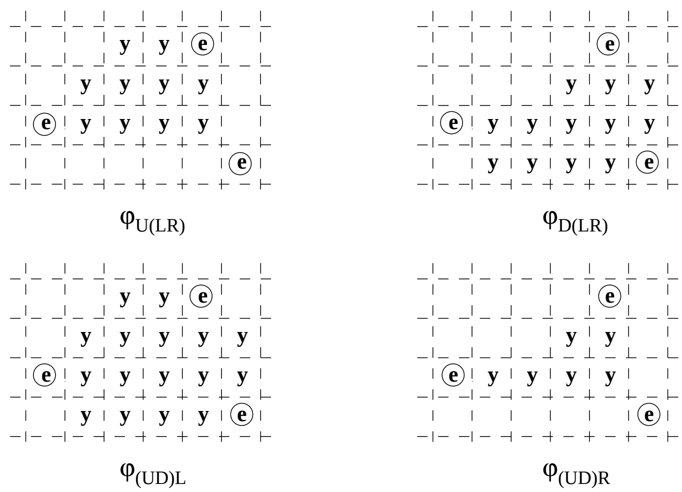


Figure 5: Four closed, convex configurations generated by the same extreme elements

not symmetric.

This asymmetry is reinforced by examining the "geodesic" paths illustrated in Figure 6. Depending on the chosen closure operator, the geodesic between the pair of extreme elements is denoted by either the "path" of x's or y's.

Actually, neither the concept of "geodesic" nor "path" has been defined as yet. A **path** between p and q of length n is a set $(pq).\rho$ of $n > 0$ elements satisfying the conditions:[5]

$$\begin{array}{ll} (a) & \forall y \in \rho, y \neq p \text{ or } q, |y.\eta| = 2, |p.\eta| = |q.\eta| = 1; \\ (b) & x_1, x_2 \in y.\eta \text{ implies } x_2 \notin x_1.\eta; \\ (c) & \text{if } n > 2, \text{ there exist elements } r_1, r_2 \text{ such that } (pr_1).\rho \text{ and } (r_2q).\rho \text{ are paths} \\ & \text{of lengths } n/2 \text{ and } n-(n/2) \text{ respectively, and } r_2 \in r_1.\eta. \end{array} \tag{2}$$

[5]In this definition, by the expression $y.\eta$ we really mean $y.\eta \cap \rho$.

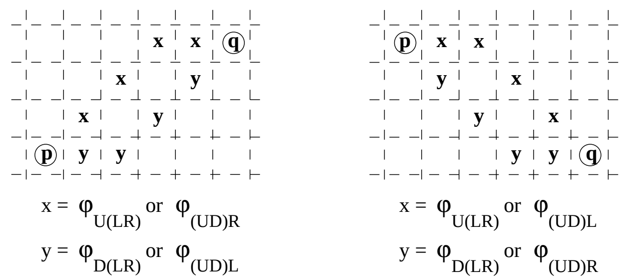


Figure 6: Geodesic paths with respect to different closure operators

Because all paths are finite, condition (c) suffices to guarantee that paths are also "connected". In this context, we can say that a set X is connected if for all $x_1, x_2 \in X$, there exists a path $(x_1x_2).\rho \subseteq X$, without this being a circular definition.

By a **uniquely generated geodesic** between p and q, denoted $(pq).\Gamma$, we mean the closed set $\{pq\}.\varphi$. We observe that since geodesics are defined with respect to particular closure operators, we should subscript Γ with respect to the appropriate distant directions, as in $(pq).\Gamma_{U(LR)}$, but we omit them whenever clarity will not suffer. Readily, $(pq).\Gamma.\kappa = \{pq\}$.

Lemma 3.2 *Every geodesic $(pq).\Gamma$ between p and q is a path $(pq).\rho$ between these elements.*

Proof: Essentially evident from the definition of $(pq).\Gamma$ as $\{pq\}.\varphi$; but we provide a more detailed argument.
(a) follows because for any $y \in (pq).\varphi, y \neq p, q$, p and q can be in at most two different distant octants, hence $|y.\eta| = 2$.
Suppose p and q are in adjacent distant octants, say $y.UR$ and $y.R$. Then $y.ur$ and $y.r \in (pq).\varphi$. Having $y \in (pq).\varphi$ contradicts minimality of closure operators, so (b) ensues.
Property (c) follows by construction. Geodesics can be determined by beginning at one extreme element, p or q depending on the chosen closure operators and their relative positions. Say we start at p. Then q is in some distant direction D and the corresponding neighbor $p.d$ becomes x_2. Iterate this procedure. The element $x_{n/2}$ satisfies the condition. □

Readily, the converse is not true.

Observe that paths and geodesics are only sets of elements. They are "thin" sets of elements because of condition (b) on paths, and they will fufill many of the roles of geometric lines in the ensuing development, but they are most definitely not lines. Also observe that by construction, any geodesic is either a "straight" vertical, horizontal or diagonal sequence of elements; or else it is a "singly bent" path beginning with a diagonal sequence elements (at one end) until some element b is reached, which we will call the **bend**, from whence it is a straight horizontal or vertical set.

Lemma 3.3 *Let* $(pq).\Gamma$ *be a uniquely generated geodesic between* p *and* q*. If* $r, s \in (pq).\Gamma$*,* $r \neq s$*, then* $(rs).\Gamma \subseteq (pq).\Gamma$*.*

Proof: We let $\varphi = \varphi_{U(LR)}$ represent a typical uniquely generated planar closure. If $q \in p.U, p.D, p.L$ or $p.R$, the lemma follows immediately.
So, *wlog* we assume that $q \in p.UL \cup p.UR$ else we can exchange the roles of p and q. Similarly, we may as well assume $q \in p.UL$ since $q \in p.UR$ will yield a symmetric argument.
Since, $r \in (pq).\Gamma$, either r is on the diagonal from p to b (the bend) or on the left horizontal from b to q. Similarly, $s \in (pq).\Gamma$ implies s is on the diagonal from p to b or left horizontal from b to q. But, wherever r and s are located, it is apparent that $(rs).\Gamma \subseteq (pq).\Gamma$ and if r and s are in different rays the two geodesics have the same bend. □

If the geodesic is straight, either horizontally, vertically, or diagonally, it is clear that it is a shortest path; but when the geodesic is bent, one might ask "is there a shorter, more direct path between p and q"? It is not immediately evident that these geodesics defined as the closure of two elements must be "shortest".

Lemma 3.4 *Let* (i, j) *and* (k, m) *be the position coordinates of* p, q *respectively. Then* $|(pq).\Gamma| = |(pq).\varphi| = max(|k - i|, |m - j|)$*.*

Proof: If p and q are aligned either horizontally or vertically the result is immediate.
Wlog we assume $q \in p.UR$ and $\varphi = \varphi_{U(LR)}$. If $q \in p.UR$ then $i > k, j < m$, and $x_1 = p.ur \in (pq).\Gamma$. Readily, x_1 has position coordinates $(i - 1, j + 1)$. If $q \in x.UR$ then $x_2 = x.ur \in (pq).\Gamma$ with coordinates $(i-2, k+2)$. This construction is iterated until $x_n = x_{n-1}.ur$ is attained with coordinates $(i - n, j + n)$ and either $i - n = k$ or $j + n = m$.
Suppose the former, then $q \in x_n.R$ and $m - (j + n)$ more horizontal elements are needed to reach q — for $m - j$ elements in all. If the latter is true, $i - (n + k)$ vertical elements are required for $i - k$ elements in all.
Proof for the other relative p, q positions and other closure operators is similar. □

Corollary 3.5 *Let* $(pq).\Gamma$ *of length* m *be the geodesic between* p *and* q *and let* $(pq).\rho$ *be any path of length* n*, then* $m \leq n$*.*

In the following development we let any neighborhood or distant direction expression, such as ur or UR denote the logical assertions $y.ur \in Y.\varphi$ or $\exists z \in y.UR \wedge z \in Y.\varphi$. With this shorthand, we can characterize those elements $y \in Y.\varphi$ that are *not* generating elements. These are the elements which *must* be in the closure. Their membership can be inferred because they are "between" two elements of the closure. Assuming that $\varphi = \varphi_{U(LR)}$, then by its definition, we have the equivalence:

$$
\begin{aligned}
&(dl \wedge ur) \vee (d \wedge u) \vee (dr \wedge ul) \vee (l \wedge r) \vee (dl \wedge r) \vee (dr \wedge l) \\
&\Leftrightarrow y \in Y.\varphi \wedge y \notin Y.\kappa
\end{aligned} \tag{3}
$$

The first four disjuncts simply assert that an element bracketed by closure elements in the vertical, horizontal or diagonal directions must itself be in the closure, but cannot be a generating (extreme) element. The next two disjuncts are unique to the $\varphi_{U(LR)}$ closure.[6] Based on (3), we can assert

Lemma 3.6 *The membership of y in a closed set $Y.\varphi$ can be determined by a neighborhood operation that examines at most 6 neighbors.*

Because of the implications (1), the neighborhood directions of (3) can be replaced with distant directions, yielding

$$
\begin{aligned}
&(DL \wedge UR) \vee (D \wedge U) \vee (DR \wedge UL) \vee (L \wedge R) \vee (DL \wedge R) \vee (DR \wedge L) \\
&\Leftrightarrow y \in Y.\varphi \wedge y \notin Y.\kappa
\end{aligned} \tag{4}
$$

and consequently

Lemma 3.7 *Given $Y.\kappa$, the generating elements of a closed set $Y.\varphi$, one can effectively determine if $y \in Y.\varphi$.*

Proof: For any element y, one need only test with respect to each of the $|Y.\kappa|$ generators for membership in 6 distant directions, which require at most two integer comparisons of position coordinates. □

Of equal interest is the negation of both sides of (3) which, after a bit of logical manipulation, yields (for $\varphi_{(U(LR))}$)

$$
\begin{aligned}
&(\neg d \vee \neg u) \wedge (\neg \vee \neg r) \wedge (\neg dl \vee (\neg r \wedge \neg ur)) \wedge (\neg dr \vee (\neg l \wedge \neg ul)) \\
&\Leftrightarrow y \notin Y.\varphi \vee y \in Y.\kappa
\end{aligned} \tag{5}
$$

This provides a way of identifying those elements of $Y.\varphi$ which are generating (extreme) elements.

[6] For $\varphi_{D(LR)}$, the two distinctive disjuncts would be $(ul \wedge r) \vee (ur \wedge l)$. For $\varphi_{(UD)R}$, we would substitute $(ur \wedge l) \vee (dr \wedge l)$, *etc.*

Let Y be any set of elements. We say $y \in Y$ is a **boundary** element if there exists some neighbor $y.u, y.ul, \ldots, y.r$ or $y.ur$ that is not in Y.[7] Because of (b) in the definition of a path (2), every element of a path is a boundary element (between the "path" and "non-path" sets).

Theorem 3.8 *If a set Y is closed with generators $Y.\kappa = \{x_1, x_2, \ldots, x_n\}$ enumerated in clockwise sequence, then its boundary consists of the geodesics $(x_i x_{i+1}).\Gamma, 1 \leq i < n$ and $(x_n x_1).\Gamma$.*

Proof: Since $\{x_i x_{i+1}\} \subseteq Y, (x_i x_{i+1}).\Gamma = (x_i x_{i+1}).\varphi \subseteq Y.\varphi$. (Alternatively, one can observe that a convex closure of Y must contain all geodesics between any pair of elements of Y.)
Suppose $(x_i x_{i+1}).\Gamma$ is not a boundary, that is there exists $y \in (x_i x_{i+1}).\Gamma$ for which every neighbor $y.d$ is in Y. Since $(x_i x_{i+1}).\Gamma$ is "thin", there exists at least one neighbor of y for which $y.d \notin (x_i x_{i+1}).\Gamma$, call it z_1.
Either $z_1.\eta \subseteq Y.\varphi$ or not. If $z_1.\eta \subseteq Y.\varphi$ choose z_2 in the direction reciprocal to y, and iterate to attain z_k. Eventually for some $z_k, 1 \leq k \leq n$ $z_k.\eta \not\subseteq Y.\varphi$. It is a boundary element. It does not satisfy the left side of equivalence (3) else it would be in $(pq).\varphi$. Hence by (5), $z_k \in Y.\kappa$ contradicting the assumed clockwise enumeration of generating elements. □

Theorem 3.9 *If $(pq).\Gamma \cap (rs).\Gamma \neq \emptyset$ then either*
(a) $(pq).\Gamma \cap (rs).\Gamma = t$, a singleton element, or
(b) $(pq).\Gamma \cap (rs).\Gamma = (bt).\Gamma$, where b is a bend and $t = p, q, r,$ s or b'.

Proof: Let $t \in (pq).\Gamma \cap (rs).\Gamma \neq \emptyset$. Assume (a) is not true; we must show that (b) is true. Suppose there exists $u \in (pq).\Gamma \cap (rs).\Gamma \neq t$. By Lemma 3.3, $t, u \in (pq).\Gamma$ implies $(tu).\Gamma \subseteq (pq).\Gamma$ and similarly $(tu).\Gamma \subseteq (rs).\Gamma$, so $(tu).\Gamma \subseteq (pq).\Gamma \cap (rs).\Gamma$.
We need only show that t (or u) must be b and u (or t) must be $p, q, r,$ s or b'. Let D denote any distant direction and d its neighboring direction. Let $u \in t.D$. We consider $y = u.d^{-1}$. If $y \in (pq).\Gamma \cap (rs).\Gamma$, we let $u = y$ and continue. Since $(pq).\Gamma \cap (rs).\Gamma$ is finite, eventually $y = u.d^{-1} \notin (pq).\Gamma \cap (rs).\Gamma$. If $y \notin (pq).\Gamma$, then either $u \in (pq).\Gamma.\kappa = \{pq\}$ or $u = b$, the bend of $(pq).\Gamma$, otherwise if $y \notin (rs).\Gamma$, then either $u \in (rs).\Gamma.\kappa = \{rs\}$ or $u = b$, the bend of $(rs).\Gamma$. In either case, u is one of p, q, r, s or b. We now consider $t \in u.D^{-1}$ and let $y = t.d$. Precisely the same reasoning will establish that t is one of p, q, r, s or b. □

In a sense this theorem is a corollary of Theorem 3.1, but not quite (see footnote below). It is fundamental to our development because in a geometry, we expect every point to be closed. In [8], Jamison calls this the S_1 *separation* axiom; in [15], it is called an *atomic*

[7] There have been extensive discussion as to whether the diagonal elements should, or should not, be regarded as neighbors of an element [17].

closure space because it gives rise to an atomic closure lattice.[8] We also expect every finite line segment to be closed. Geodesics, $(pq).\Gamma$, have been defined precisely to insure this. In any closure space, the intersection of closed sets must be closed. Now suppose that $(pq).\Gamma \cap (rs).\Gamma$ were two "separated" elements t and u. This closed set $\{tu\}$ must be simple, *i.e.* generated by itself, or $\{tu\}.\kappa = \{tu\}$. But in a geometry, any two distinct points must also generate a line, or equivalently $(tu).\Gamma.\kappa = \{tu\}$, which is impossible. Several promising closure operators in the plane have been discarded because the intersection of two closed sets was separated into two distinct subsets contradicting any form of unique generation.

4 Geodesic Extensions and Half Planes

We have demonstrated that the boundary of any closed set is comprised of geodesics between its generating extreme points. Now we wish to show that the intersection of 3, or more, independent half planes must be closed. First, we must define what we mean by a half plane. In Euclidean geometry, any unbounded straight line separates the plane into two half planes, each of which is regarded as convex. Any two points on this straight line, or a point and a slope, are sufficient to define the half planes.

In Section 3, we used direction implications to define $\varphi_{U(LR)}$ and other uniquely generated closure operators. The same implications defined the notion of a geodesic between two points p and q. We might extend $(pq).\Gamma$ indefinitely by repeatedly replicating it as $(p'q').\Gamma$ and linearly translating it so that $p' = q$. The geodesic of Figure 6 has been so replicated in Figure 7. The geodesic $(rr').\Gamma$ indicated by the dashes is not contained in the path $(pq').\rho$,

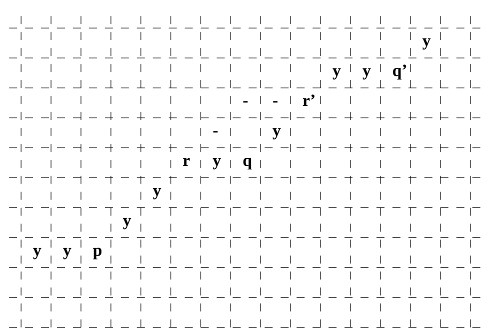


Figure 7: A geodesic extension that is not geodesic

so by Lemma 3.3, $(pq').\rho$ is not a geodesic.

[8]There exist non-atomic uniquely generated closure spaces. The left, or right, ideals of a partial order constitute one example.

Consequently, we must instead employ a somewhat counter-intuitive iterative construction. A geodesic $(p_{i+1}q_{i+1}).\Gamma$ is said to be an **extension** of the geodesic $(p_iq_i).\Gamma$ if for the unique (by condition (a)) neighbor x of $p_i \in (p_iq_i).\Gamma$, p_{i+1} is its reciprocal neighbor and q_{i+1} is the reciprocal neighbor of $q_i \in (p_iq_i).\Gamma$. The indefinite *extension* of $(pq).\Gamma$, denoted $(pq).\Gamma.\epsilon$, is $(p_nq_n).\Gamma$, where $p_0 = p$ and $q_0 = q$ and n can be arbitrarily large as indicated in Figure 8. Note that Lemma 3.3 still holds, but $(rs).\Gamma \subseteq (pq).\Gamma$ does not imply that

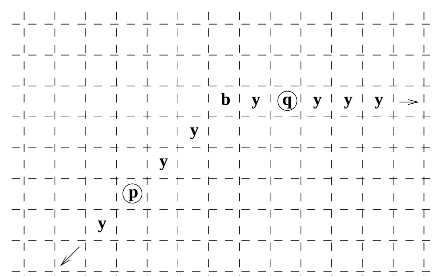


Figure 8: A geodesic extension, $(pq).\Gamma.\epsilon$

$(rs).\Gamma.\epsilon \subseteq (pq).\Gamma.\epsilon$ as illustrated by Figure 9. The extension of $(r_1s_1).\Gamma$ is indicated by 1's;

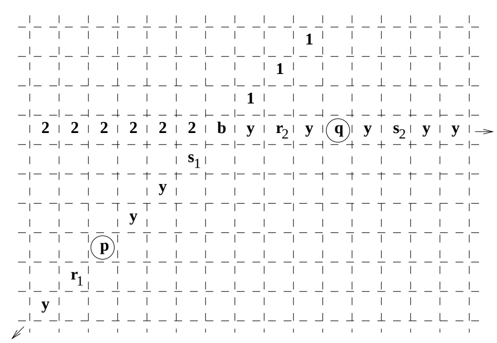


Figure 9: $(rs).\Gamma \subseteq (pq).\Gamma$ does not imply $(rs).\Gamma.\epsilon \subseteq (pq).\Gamma.\epsilon$

$(r_2s_2).\Gamma.\epsilon$ is indicated by 2's. Readily two non-straight geodesics have the same extension if and only if both have the same *bend*.

By a **ray** $(bq).\epsilon$, we mean just one arm of a bent geodesic extension. In Figure 8, the sequence $< b\ y\ q \ldots >$ begins a horizontal ray $(bq).\epsilon$ of $(pq).\Gamma.\epsilon$ and $< b\ y\ y\ p \ldots >$ are the initial elements of a diagonal ray $(bp).\epsilon$. Straight horizontal or vertical or diagonal geodesics have no constituent rays.

We observe that neither geodesic extensions nor rays are regarded as closed sets; if they were they would not be uniquely generated.

Theorem 4.1 *If Y is a closed subset, then Y is the intersection of n half planes $H_i, 1 \leq i \leq n$ where $n \geq 3$.*

Proof: Let Y be generated by $Y.\kappa = < x_1, x_2, \ldots x_n >$. By Theorem 3.8, Y is bounded by the geodesic segments $(x_i x_{i+1}).\Gamma$. Extending these creates the n half planes whose intersection is Y. □

In Euclidean E^2, any 3 non-parallel lines must intersect in a triangle,[9] and the intersection of some 3 of the 6 half planes they define must constitute this closed, convex triangle. Such is not the case in a discrete rectilinear space. One can have 3 non-parallel geodesics that need not constitute the boundaries of a 3-gon, as in Figure 10(a). One may have two

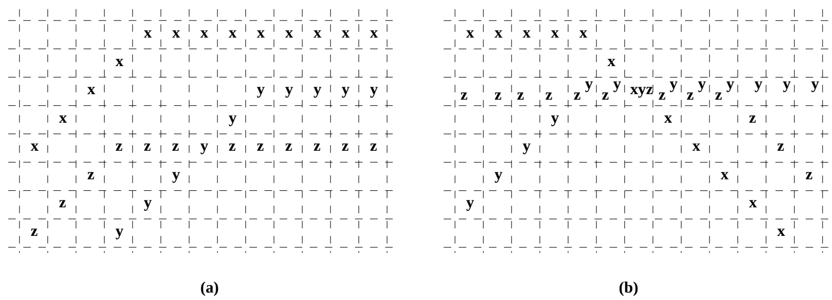


Figure 10: Anomalies arising in $\varphi_{U(LR)}$

geodesics that are each parallel to a third, but themselves are intersecting, as in Figure 10(b). By **parallel** we mean that the extensions have no finite intersection. (It is not clear whether this definition should specify "no intersection" or "no finite intersection". We have chosen the latter.)

It is fairly clear from Figures 10(a) and (b) that it is bent geodesics that can create anomalies. With respect to $\varphi_{U(LR)}$, which we have been using for our running examples, all bent (non-straight) geodesics, $(pq).\Gamma$, begin by initially adding elements above and to the left or right on a diagonal from p. Then elements are added in a vertical (up), or horizontal (left or right) direction until reaching q. These four possible patterns are illustrated in Figure 11. The orientation of the bent geodesic can be identified by one of the three character direction strings, URR, URU, ULL or ULU.

[9] A single point is regarded as a degenerate triangle.

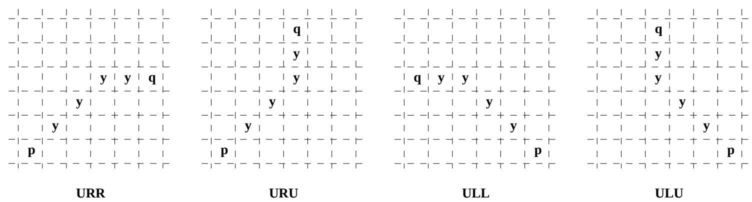


Figure 11: The four bent geodesic orientations of $\varphi_{U(LR)}$

Theorem 4.2 *Let $g_1 = (p_1q_1).\Gamma, g_2 = (p_2q_2).\Gamma$ and $g_3 = (p_3q_3).\Gamma$ be three geodesics, no two of which are parallel. If*

(a) all three geodesics are straight (vertical, horizontal, diagonal), or

(b) all bent geodesics have the same orientation,

then either g_1, g_2 and g_3 define a 3-gon or they intersect in a single closed geodesic $(tu).\Gamma$ which may be a single point.

Proof: (a) Assume none of g_1, g_2 or g_3 is bent. Suppose g_1 is horizontal (or vertical). Then neither g_2 nor g_3 can be horizontal (or vertical) and still intersect g_1. At least one, say g_2, must be diagonal, because both cannot be vertical (or horizontal) and intersect each other. So g_3 can be aligned with either the other diagonal direction or vertical (or horizontal).

In any case, the three straight geodesics must form a 3-gon or intersect in a common element.

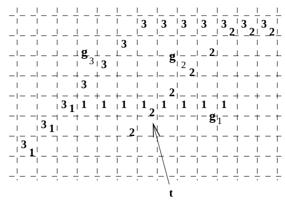
(b) Assume g_1 is bent. If g_2 and g_3 are straight, the conditions of (a) are effectively met with g_1 taking the role of the diagonal g_2 in that argument, yielding the same result.

So we can assume that both g_1 and g_2 are bent, and that both have the same orientation. To provide a more concrete argument, we will assume *wlog* that both are URR as is Figures 10(a) and 11(a). (Other orientations would be handled similarly.) Since $g_1 \cap g_2 \neq \emptyset$, by Theorem 3.9 either $g_1 \cap g_2 = \{r\}$ (a single element as in Figure 10(a)) or $g_1 \cap g_2$ is a subset of a single horizontal or diagonal ray.

We consider the latter case first; $g_1 \cap g_2$ is a subset of a horizontal or diagonal ray. If g_3 intersects g_1 and g_2 in this subset, $g_1 \cap g_2 \cap g_3$ is either a single element or a single interval. If $g_3 \cap g_1 = \{r\} \neq \{s\} = g_3 \cap g_2$, then let t be that element of $q_1 \cap q_2$ such that $(rt).\Gamma \cap (st).\Gamma = \{t\}$. (*I.e.* t is an extreme element of $g_1 \cap g_2$.) The 3-gon with extreme elements r, s and t is defined by g_1, g_2 and g_3.

If $g_1 \cap g_2$ is a single element $\{t\}$ (the former case), then g_3 can intersect g_1 and g_2 either in distinct singleton elements, in which case the result follows easily, or in short geodesic intervals. If $g_1 \cap g_3$ (or $g_2 \cap g_3$) is a single element, then the preceding argument (with at least two intersections as singleton elements) holds. So suppose $g_1 \cap g_3$ is a geodesic interval, say *wlog* diagonal. If $g_2 \cap g_3$ is diagonal, then g_3 is diagonally "straight" and $g_1 \cap g_2$ must be a diagonal interval as well, contradicting the assumptions of this case. So $g_2 \cap g_3$ must be horizontal. This yields a configuration as in the figure

The elements $b \in g_1, b' \in g_2$ and t define the 3-gon. □



In Figure 12(a), 5 elements are arranged on a rectilinear grid. Some of the geodesics between them are indicated by dashed lines. These delimit closed subsets of which the elements are generators. The induced closure lattice, $\mathcal{L}$, is shown to the right. Most of its

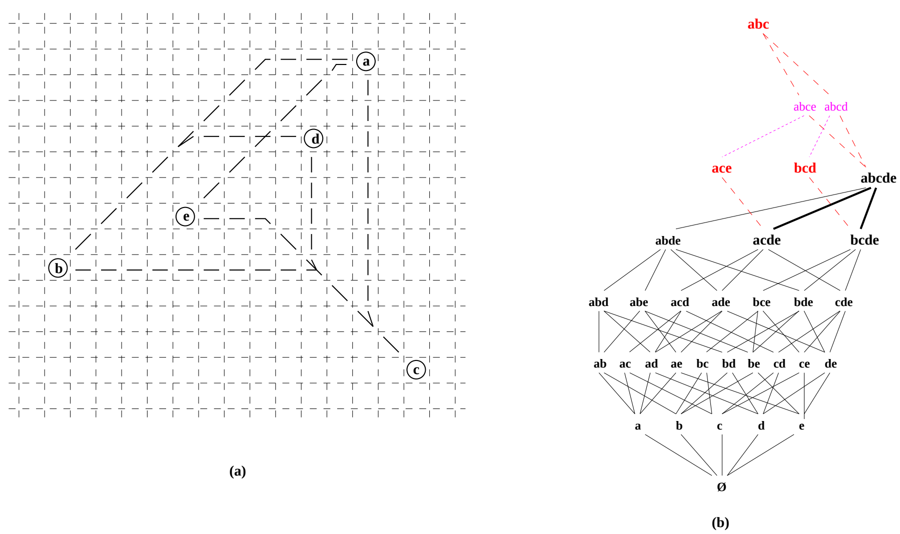


Figure 12: Five points, their closures using $\varphi_{U(LR)}$ and closure lattice, $\mathcal{L}$

$2^5 = 32$ subsets are closed. It is worthwhile to expand this lattice by inserting another point f in the closed set ade. Depending on its placement, either d, e, de, or neither may be in $abf.\varphi$. Observe that this space is only an enlargement of Figure 5.

The reason antimatroid closure spaces are of interest in computational procedures lies in a fundamental characteristic. Given any closed set Y of interest, removal of a generating element e must yield another closed set. This provides an effective way of exploring the large, non-simple[10] sets of interest by a systematic shelling process that removes generators. Greedy algorithms employ such a process that systematically reduce a problem to a smaller

[10] A set Y is **simple** if $Y.\kappa = Y.\varphi$.

one. Shelling processes, called greedoids, have been thoroughly explored in [9]. But, one must be careful. They shell "feasible" sets which are the complement of our "closed" sets.

Clearly we have been developing a kind of geometry in which singleton elements correspond to points, geodesic sets correspond to straight lines, and closed sets correspond to convex regions. Many geometric properties are preserved, such as the fact that the boundary of a convex region (closed set) is a polygon of straight lines (geodesics) and that two straight lines (geodesics) cannot meet in two distinct points. But, there are some interesting differences as well. While two lines cannot intersect in distinct points, they can coincide over a segment. And two straight lines which are parallel to a third straight line (*i.e.* never intersect it) may themselves intersect.

5 Other Closure Patterns

In the preceding two sections we have developed a discrete geometry based on the closures $\varphi_{U(LR)}$, $\varphi_{D(LR)}$, $\varphi_{L(UD)}$, $\varphi_{R(UD)}$, which employ diagonal sequences of elements. We can restrict these closure to only horizontal and vertical sequences by modifying the closure implications (1) For example, the closure $\varphi_{U(LR)}$ is replaced by $\varphi_{(UL)(RU)}$ defined by the implications (a), (b), (c), (d) together with

(e) $z \in y.UL$ implies $y.u \in Y.\varphi$

(f) $z \in y.UR$ implies $y.r \in Y.\varphi$.

Note that we have only replaced $y.ul$ and $y.ur$ in the consequent of these two implications by $y.u$ and $y.r$. Notice that here we are effectively ordering the steps taken in generating the geodesic closure. The closure $\varphi_{U(LR)}$ could be interpreted as moving *up and (left or right)* for directions not horizontal or vertical; whereas $\varphi_{(UL)(RU)}$ can be interpreted as *(up and left) or (right and up)* for these directions. One is tempted to apply a form of distributivity to the first, commutivity to the second, and call them equivalent. But they are not. To see this consider Figure 13, in which the closed rectangle can be generated by the two lower

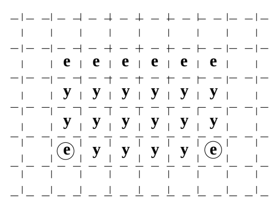


Figure 13: Any upper element is extreme wrt. $\varphi_{U(LR)}$

extreme elements plus any third chosed from the top row. "Moving right" before "up" is crucial in a city block closure.

Applying this closure concept yields the geodesics of Figure 14

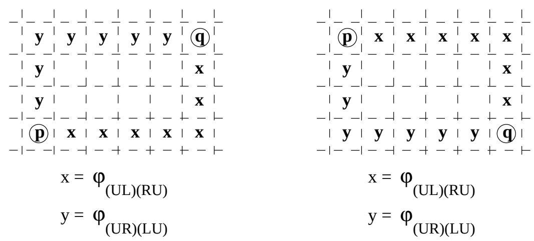
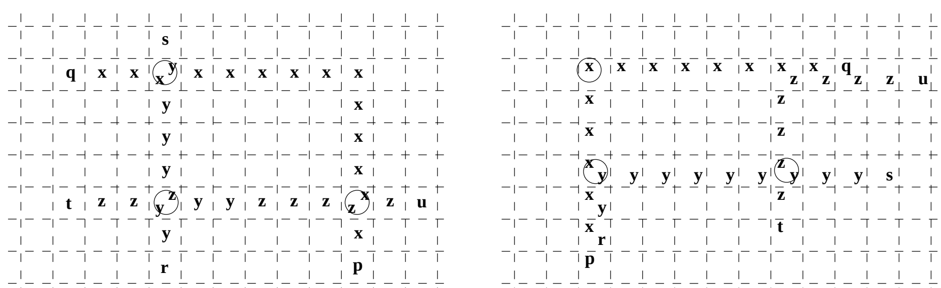


Figure 14: Geodesics, $(pq).\Gamma = (pq).\varphi$

The results of the preceding sections can all be restated and reproven for this kind of closure (?we hope?), with arguments that replace diagonal geodesics with “ordered” geodesics.

Some interesting consequences ensue. Consider the 3-gons of Figures 15(a) and (b). Three sided figures “look” to be rectangular. But, the three generating (extreme) elements



(a) **(b)**

Figure 15: Geodesics, $(pq).\Gamma = (pq).\varphi$

have been circled for emphasis. This appears to be counter-intuitive; however for VLSI and other rectangularly organized systems it can be valuable. In the simplicial development of any plane geometry, one has points, lines, and 3-gons (or 2-simplexes). These are the building blocks. If rectangular blocks are the simplest 2-dimensional shapes, then these should be the system’s 2-simplexes. That 3-gons are rectangular figures opens up the possibility of a simplicial development of circuit board layout and design.

The reader can verify that the same techniques can be used to define a uniquely generated closure on hexagonal grids. The direction operators must be redefined to accomodate 3, rather than 4, principle directions. Perhaps they should be radially numbered rather than lettered. But, in any case the same kinds of results follow.

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