On a Constrained Bin-packing Problem

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Abstract

We study a bin-packing problem which is one-dimensional and is constrained in the manner items are placed into bins. The problem is motivated by a practical real-time scheduling problem, where redundant periodic tasks need to be assigned to a multiprocessor system. The problem is stated in the traditional light: to use as few bins as possible to pack a given list of items, and it is a generalization of the classical bin-packing problem. We first propose a heuristic algorithm called First-Fit-K to solve the bin-packing problem, and then prove that First-Fit-K has an asymptotical worst-case tight bound of 1.7. We also study the average-case performance of the algorithm. The simulation results show that First-Fit-K performs within 10% of the optimal solution.

Index Terms: Bin-packing, combinatorial optimization, algorithm design and analysis, simulation, real-time scheduling, EDF

I. Introduction

In this paper, we study the following *bin-packing problem*: A list of items are to be packed into a potentially infinite number of unit-size bins. Each item has a color and a size which is no more than one. For each color, there are at most $\kappa \geq 1$ items. Then given a list of colorful items, what is the minimum number of bins that is required to pack the items such that no two items with the same color are packed to a bin?

This problem is a natural generalization of the one-dimensional bin-packing problem studied in [7]. Indeed, if the number of items with the same color is one, i.e., $\kappa = 1$, then the two problems coincide. Although there have been many extensions or variations to the classical bin-packing problem, this new problem has not been studied yet in the literature according to the best of our knowledge. The classical bin-packing problem is known to be NP-hard, from which it follows trivially that the new bin-packing problem is also NP-hard. For this reason we shall focus on fast heuristic algorithms for solving this problem, seeking to prove close bounds on the extent to which they can deviate from optimality. Due to the complexity involved, the analysis of these simple approximation methods represents a permanent challenge (see, e.g., Coffman et al [2])

It is a common practice to analyze the performance ratio of the algorithm under study when working with approximation algorithms for combinatorial optimization problems [5]. Let $A(I)$ denote the performance of a given algorithm for an instance I of a particular combinatorial optimization problem and let $OPT(I)$ denote the performance of an optimal algorithm for the same instance. The ratio of $A(I)$ to $OPT(I)$, considered over all instances I, provides us with an indicator of the quality of the given algorithm. To be specific for bin-packing heuristics, let $N_A(L)$ and $N_0(L)$ (or N_0) denote the number of bins required by the heuristic A and the optimal number of bins required to schedule a given list L of items, respectively. Then, the asymptotical worst-case bound for heuristic A is determined by

$$\Re_A^\infty = \inf\left\{ r \geq 1 : \text{for some } N \in Z^+, N_0 \geq N, \frac{N_A(L)}{N_0} \leq r \text{ for all lists } L \right\}$$

It is apparent that the smaller the $\Re_A^\infty$'s value is, the better the heuristic algorithm A performs in terms of the worst-case scenario. In other words, the smaller the $\Re_A^\infty$'s value is, the closer the heuristic solution is to the optimal one. Hence, we want to minimize $\Re_A^\infty$ as much as possible when we design a heuristic algorithm.

Many heuristic algorithms, such as Next-Fit, First-Fit [6, 7], and Harmonic Fit [9], have been devised to solve the classical bin-packing problems and other bin-packing problems. Among the different strategies, the First-Fit strategy has been frequently adapted to solve the various bin-packing problems and is one of the best studied ones. The First-Fit strategy is a simple, on-line one, and yet it can deliver near-optimal performance. For the classical bin-packing problem, the First-Fit heuristic has a tight bound of 1.7, while no on-line algorithm can have an asymptotical worst-case bound less than 1.53 [1, 10]. By ordering the items according to their decreasing sizes and applying the First-Fit strategy to pack the new list of items, we have the famous First-Fit-Decreasing (or FFD) heuristic, which is clearly off-line and has a tight bound of 11/9. Heuristic algorithms with polynomial time complexity can also be devised that have an upper bound arbitrarily close to one [3, 8], i.e., for any $\varepsilon > 0$, there is an algorithm A_ε, whose running time grows polynomial with $1/\varepsilon$, that has an asymptotic bound of $1 + \varepsilon$. But these algorithms are generally too complicated to be of practical applications. For these reasons, it is interesting to study the performance of the First-Fit-K heuristic, which is a natural adaptation from the First-Fit strategy.

We first came upon this bin-packing problem when we were investigating the issues of supporting fault-tolerance in a real-time system [12]. A scheduling problem arises in a situation where, for fault-tolerance purposes, multiple versions are used for each periodic task so that versions of a task must be executed on different processors. Specifically, we were dealing with the following scheduling problem: a set of n tasks $\Sigma = \{\tau_1, \tau_2, \ldots, \tau_n\}$ is given, where $\tau_i = ((C_{i,1}, C_{i,2}, \ldots, C_{i,\kappa_i}), R_i, T_i)$ for $i = 1, 2, \ldots, n$, and $(C_{i,1}, C_{i,2}, \ldots, C_{i,\kappa_i})$ are the computation times of the κ_i versions of task τ_i. R_i and T_i are the release time and period of task τ_i, respectively. The deadline of each request of a task is the arrival of its next request. What is the minimum number of processors required to execute the task set such that versions of a task are executed on different processors and all task deadlines are met by the *Earliest Deadline First* (or EDF) algorithm?

Liu and Layland proved that a set of periodic tasks can be feasibly scheduled by EDF if and only if $\sum_{i=1}^{n} C_i/T_i \le 1$, and the release time of each task, R_i, does not affect the schedulability of a set of periodic tasks [11]. Since $0 < C_i/T_i \le 1$ and $\sum_{i=1}^{n} C_i/T_i \le 1$, we can treat the assignment of a set of tasks to a single processor as packing a list of items into a bin with a unit size. The quantity $u_i = C_i/T_i$ for a task (version) corresponds to the size of an item. In order to distinguish versions belonging to one task from those belonging to another, we assign colors to them such that versions belonging to one task share the same color. Versions belonging to different tasks have different colors. Then items with the same color cannot be assigned to the same bin and the maximum number of items with the same color is $\kappa = \max_{1 \le i \le n} \kappa_i$. The number of colors therefore corresponds to the number of tasks in a task set. Hence, the scheduling problem is equivalent to the above bin-packing problem.

Besides the scheduling problem described above, the bin-packing problem also occurs in a number of other applications. For example, the problem of allocating a set of parallelized tasks to the minimum number of processors such that the completion time of the whole schedule is bounded can also be reduced to this bin-packing problem.

We will present our heuristic algorithm First-Fit-K and analyze its asymptotical worst-case performance in Section II. An empirical study through simulation on the average-case performance of the algorithm appears at the end of Section II. We conclude in Section III with a look at future research directions.

II. The Design and Analysis of First-Fit-K

The design of First-Fit-K is quite straightforward: to ensure that no two items with the same color is assigned to the same bin, we only need to make sure that the bin that is selected by First-Fit does not contain an item with the same color. The algorithm is given as follows:

First-Fit-K (or **FF-K**): Let the bins be indexed as B_1, B_2, …, with each initially filled to level zero. Given a list of colorful items, where the size of each item is no more than 1 and the maximum number of items with the same color is κ, the items are assigned to bins in the order they are given. In assigning an item to a bin, the smallest-indexed bin that does not contain an item with the same color as the item being assigned and in which the item can be fit, is selected to contain the item. An item is assigned to a new bin if it cannot be assigned to any non-empty bin.

The main result is stated in the following theorem. Where there is no confusion, we refer an item of size b simply as item b.

Theorem 1: *For any list L of items b_1, b_2, …, b_n, FF-K(L) $\le 1.7L^* + 2.19\kappa$, where κ is the maximum number of items with the same color, FF-K(L) is the number of bins used by FF-K to pack the list L, and L^* is the minimum number of bins used to pack the same list.*

Before proving the theorem, we need to establish several lemmas.

Lemma 1: *Suppose the maximum number of items with the same color is κ. Among all the bins to each of which $n \ge c \ge 1$ items are assigned, there are at most κ of them, each of which is no more than $c / (c + 1)$ full.*

Proof: The lemma is proven by contradiction. Suppose that there are $\kappa + 1$ bins each of which is no more than $c / (c + 1)$ full. Let $B_1, B_2, \ldots, B_{\kappa + 1}$ be such $\kappa + 1$ bins and $b_{i,j}$ be the jth item that is assigned to bin B_i, for $1 \le i \le \kappa + 1$ and $1 \le j \le n$. Then $\sum_{j=1}^{n} b_{i,j} \le c / (c + 1)$, for $1 \le i \le \kappa + 1$.

Let us look at the sizes of items assigned to the last bin, $B_{\kappa + 1}$, among the $\kappa + 1$ bins. Since there are $n \ge c$ items in the bin $B_{\kappa + 1}$ and $\sum_{j=1}^{n} b_{\kappa+1,j} \le c / (c + 1)$, there must exist an item $b_{\kappa+1,z}$ in the bin $B_{\kappa + 1}$ such that $b_{\kappa+1,z} \le 1 / (c + 1)$ and $z \in \{1, 2, \ldots, n\}$. If not, then $\sum_{j=1}^{n} b_{\kappa+1,j} > c / (c + 1)$.

Since $\sum_{j=1}^{n} b_{i,j} + b_{\kappa+1,z} \le 1 / (c + 1) + c / (c + 1) = 1$ for $1 \le i \le \kappa$, and $b_{\kappa+1,z}$ cannot be assigned to the bin B_i, there must exist one and only one item $b_{i,j}$ among the items $\{b_{i,j} | j = 1, 2, \ldots, n\}$, that has the same color as $b_{\kappa+1,z}$ does, for all $i = 1, 2, \ldots, \kappa$. In other words, there are a total of $\kappa + 1$ items with the same color as item $b_{\kappa+1,z}$. This is a contradiction to the assumption that the maximum number of items with the same color is κ. Therefore the lemma must be true. ■

In the following, we define a weighting function W(α) that maps the size of an item, α, to a number between zero and one, i.e., $W(\alpha)$: $(0, 1] \rightarrow (0, 1]$, as given in Figure 1. We call the value of $W(\alpha)$ the weight of item α, and the sum of the weights of the items assigned to a bin the weight of the bin. The weighting function is defined in such a way that with a few exceptions, the weight of a bin in the completed FF-K packing is equal to or greater than 1, and the weight of a bin in the optimal packing is no greater than 1.7. Note that, although this weighting function was first used by Johnson et al [7] in deriving the bound for First-Fit, the proof here follows a different route and is a little bit more involved, due to the additional color constraint placed upon the placement of items.

$$W(\alpha) = \begin{cases} (6\alpha)/5 & 0 < \alpha \le 1/6 \\ (9\alpha)/5 - 1/10 & 1/6 < \alpha \le 1/3 \\ (6\alpha)/5 + 1/10 & 1/3 < \alpha \le 1/2 \\ 1 & 1/2 < \alpha \le 1 \end{cases}$$

We first claim that for any bin in the optimal packing, the total weight of the bin is no greater than 1.7, i.e., $\sum_{i=1}^{m} W(b_i) \le 1.7$.

Lemma 2: *Let a bin be filled with items $b_1, b_2, \ldots, b_m$. Then* $\sum_{i=1}^{m} W(b_i) \le 1.7$.

This lemma was readily proven by Johnson et al in [7].

In order to prove that, with a limited number of bins, the weight of each bin in the completed FF-K packing is no less than one, we divide the bins into several groups according to the levels

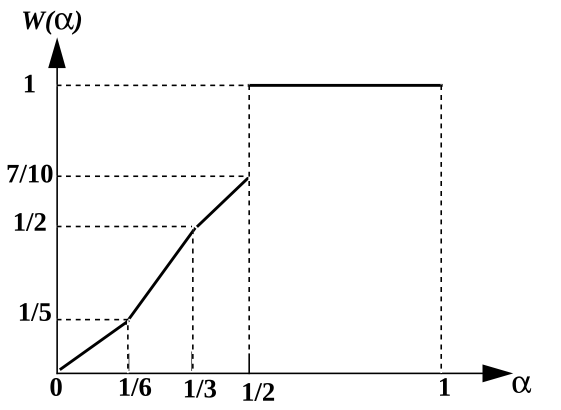

Figure 1: Weighting Function $W(\alpha)$

they are filled to. Since a bin can be filled to a level from zero to one, we instead divide the bins into groups according to the regions their levels fall into. A total number of seven regions is defined: (0, 1/2], (1/2, 2/3], (2/3, 2/3 + 1/18), [2/3 + 1/18, 3/4), [3/4, 4/5), [4/5, 5/6), and [5/6, 1]. For each region, the result is stated in a lemma. The proof of the theorem is given at the end.

Lemma 3: *Let a bin be filled with items $b_1 \geq b_2 \geq \ldots \geq b_m$. If $\sum_{i=1}^{m} b_i \leq 1/2$, then there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$ and $m \geq 1$.*

Proof: According to Lemma 1, among all bins to each of which $m \geq 1$ items are assigned, there are at most κ of them, each of which is no more than 1/2 full. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$. ■

Lemma 4: *Let a bin be filled with items $b_1 \geq b_2 \geq \ldots \geq b_m$. If $1/2 < \sum_{i=1}^{m} b_i \leq 2/3$, then there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$ and $m \geq 2$.*

Proof: For $1/2 < \sum_{i=1}^{m} b_i \leq 2/3$, the bins with $\sum_{i=1}^{m} W(b_i) < 1$ must be assigned at least two items, i.e., $m \geq 2$. If $m = 1$, then $b_1 > 1/2$ and $\sum_{i=1}^{m} W(b_i) \geq 1$.

According to Lemma 1, among all bins to each of which $m \geq 2$ items are assigned, there are at most κ of them, each of which is no more than 2/3 full. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$. ■

For the region of (2/3, 2/3 + 1/18), there may be an infinite number of bins with $\sum_{i=1}^{m} W(b_i) < 1$. However, the deficiency of weights created by these bins can be bounded, as shown by the next lemma. However, in order to show that this deficiency can indeed be bounded, we need a few definitions.

Definition 1: Let a bin B_i be filled with items $b_1, b_2, \ldots, b_m$. The color of an item b_j is denoted by $\chi(b_j)$, and the set of colors of the items in a bin B_i is denoted by $\chi(B_i)$. The deficiency δ_i of a bin B_i is defined as $\delta_i = 1 - \sum_{i=1}^{m} b_i$, i.e., where the bin is filled up to the level of $1 - \delta_i$ in the completed FF-K packing. For convenience in defining the coarseness of a bin, we introduce an imaginary bin with a zero index, such that its coarseness is zero, and its color set is empty. Then the coarseness of a bin with an index larger than zero is defined as

$$\alpha_i = \max_{\{0 \le j < i \wedge (\chi(B_j) \cap \chi(B_i)) = 0)\}} \delta_j \text{, for } i \ge 1.$$

Specifically, the coarseness of a bin is equal to the maximum deficiency, among all the bins that are ahead of the current bin and that do not share any color with the current bin. Intuitively, the size of each item in a bin must be larger than the coarseness of the bin. If a bin has a coarseness of zero, then either it is the first one or, most likely, every bin ahead of it shares at least one color with it.

Lemma 5: *Let a bin B_i with coarseness α_i be filled with items $b_1 \ge b_2 \ge \ldots \ge b_m$ and $2/3 < \sum_{i=1}^{m} b_i < 3/4$. Then there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$ and $m \ge 3$. If l is the number of bins with $2/3 < \sum_{i=1}^{m} b_i < 2/3 + 1/18$, $\sum_{i=1}^{m} W(b_i) = 1 - \beta_i$, $\beta_i > 0$, and $m = 2$, then $\sum_{i=1}^{l} W(B_i) > l - 9\kappa/20$.*

Proof: According to Lemma 1, among all bins to each of which $m \ge 3$ items are assigned, there are at most κ bins of them, each of which is no more than 3/4 full. For those bins with $m \ge 3$, there are at most κ bins with $2/3 < \sum_{i=1}^{m} b_i < 3/4$. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$.

Accordingly, we need only to focus our attention on the bins each of which is assigned two items, i.e., $m = 2$. Furthermore, $1/2 > b_1 \ge 1/3$ and $b_2 < 1/3$, since $2/3 < \sum_{i=1}^{2} b_i < 2/3 + 1/18$ and $\sum_{i=1}^{m} W(b_i) < 1$.

Claim 1: There are at most $(3\kappa)/2$ such bins that have a coarseness of zero.

Let us consider the worst case configuration of the FF-K bin-packing where the maximum number of bins with zero coarseness is achieved. Note that for these bins, a bin with zero coarseness implies that all the bins ahead of it contain one of the two colors it contains. This is because each of these bins has a deficiency of at least $1 - (2/3 + 1/18)$.

Recall that for the first of these bins, it contains exactly two colors. For the bins that follows it, every one of them must contain at least one of its colors. Now we want to find out the maximum number of bins that can possibly satisfy this constraint. Let n be the number to be derived. Then it is apparent that $n < 2\kappa$, because the maximum number of items with the same color is κ.

Let c_1 and c_2 be the two colors in the first bin. Let i_1 be the number of bins that immediately follow the first bin and share the same color c_1 and i_2 be the number of bins that immediately follow the first bin and share the color c_2. If $i_1 = i_2$, then $n \le i_1 \le \kappa$. Let us assume that $i_1 > i_2$. Let $j > 0$ be the number of bins that immediately follow the i_1th bin and have one color c_2. Then $i_2 + j \le \kappa$, since the number of bins containing color c_2 must be no more than κ. Furthermore, $i_1 - i_2 + j \le \kappa$. This is because the $(i_1 - i_2)$ bins that immediately follow the first i_2 bins must share one color with the j bins that immediately follow the i_1th bin with the other color being c_2. This is illustrated in Figure 2.

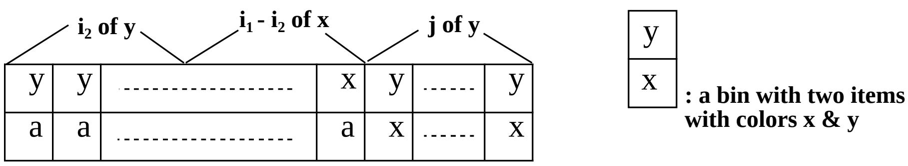

Figure 2: Worst-case Configuration of Zero Coarseness

Since $i_1 \le \kappa$, $i_1 - i_2 + j \le \kappa$, and $i_2 + j \le \kappa$, we conclude that $n \le i_1 + j \le (3\kappa)/2$.

Claim 2: $\sum_{i=1}^{2} W(b_i) \ge 1$ if $\sum_{i=1}^{2} b_i \ge 1 - \alpha_i$.

For any such bin with coarseness $\alpha_i > 0$, α_i must be larger than 1/3 – 1/18 (since $\sum_{i=1}^{2} b_i <$ 2/3 + 1/18).

Let b_1 and b_2 be the two items assigned to a bin B_i and $b_1 \ge b_2$. Then $b_1 > \alpha_i \ge 1/3 - 1/18$ and $b_2 > \alpha_i \ge 1/3 - 1/18$, according to the definition of coarseness. If $\alpha_i \ge 1/3$, then $b_2 \ge 1/3$ and $\sum_{i=1}^{2} W(b_i) \ge 1/2 + 1/2 = 1$.

If $\alpha_i < 1/3$, then $b_1 \ge 1/3$, and $b_2 < 1/3$. Otherwise, $b_1 + b_2 < 2/3$, which contradicts the assumption that $\sum_{i=1}^{2} b_i > 2/3$. Then $\sum_{i=1}^{2} W(b_i) = 6b_1/5 + 1/10 + 9b_2/5 - 1/10 > 6\sum_{i=1}^{2} b_i/5 + 3b_2/5 > (6/5) \bullet (1 - \alpha_i) + 3\alpha_i/5 = 1 + 1/5 - 3\alpha_i/5 > 1$, since $b_2 > \alpha_i$ and $\alpha_i < 1/3$.

For future reference, if $\sum_{i=1}^{2} W(b_i) = 1 - \beta_i$ and $\beta_i > 0$, then we must have $\sum_{i=1}^{2} b_i < 1 - \alpha_i$, $1/3 \le b_1 \le 1/2$, and $1/6 < b_2 < 1/3$. $\sum_{i=1}^{2} W(b_i) = 6b_1/5 + 1/10 + 9b_2/5 - 1/10 > 6b_1/5 + 9(2/3 - b_1)/5 = 6/5 - 3b_1/5 \ge 9/10$ since $b_1 \le 1/2$. In other words, $\beta_i \le 1/10$.

Claim 3: $\sum_{i=1}^{m} b_i \le 1 - \alpha_i - 5\beta_i/9$ if $\sum_{i=1}^{m} W(b_i) = 1 - \beta_i$ with $\beta_i > 0$.

To prove this claim, let b_1 and b_2 be the two items assigned to a bin B_i with $b_1 \ge b_2$. Suppose

$\sum_{i=1}^{m} b_i = 1 - \alpha_i - \gamma$ with $\gamma > 0$. Then we can construct a bin filled with two items σ_1 and σ_2 such that $\sigma_1 + \sigma_2 = b_1 + b_2 + \gamma$, and $\sigma_1 \le 1/2$ and $\sigma_2 \le 1/2$. Then $W(\sigma_1) + W(\sigma_2) \ge 1$. Since the slope of the weighting function W in the range of (0, 1/2] does not exceed 9/5, therefore $W(\sigma_1) + W(\sigma_2) \le \sum_{i=1}^{m} W(b_i) + 9\gamma/5$. In other words, $1 \le 1 - \beta_i + 9\gamma/5$. $5\beta_i/9 \le \gamma$. $\sum_{i=1}^{m} b_i \le 1 - \alpha_i - 5\beta_i/9$.

Suppose that in the completed FF-K packing, let l be the number of bins with $\sum_{i=1}^{m} W(b_i) < 1$. Among the l bins, let $B'_1, B'_2, \ldots, B'_h$ be the bins that have non-zero coarseness. If we group these bins according to $\chi(B'_i) \cap \chi(B'_j) = 0$ for any pair of bins in a group, then there are at most $(3\kappa)/2$ different groups, according to Claim 1. Within each group, let n be the number of bins in such group. Then $\alpha_i < \alpha_j$ if $i < j$. Since $\alpha_i \ge \alpha_{i-1} + 5\beta_{i-1}/9$, for $1 < i \le n$, then $\sum_{i=1}^{n-1} \beta_i \le 9/5 \bullet \sum_{i=2}^{n} (\alpha_i - \alpha_{i-1}) = 9/5 \bullet (\alpha_n - \alpha_1) \le 9/5 \bullet (2/3 + 1/18 - 2/3) = 1/10$. Since $\beta_n \le 1/10$, we have $\sum_{i=1}^{n} \beta_i \le 2/10$. Therefore $\sum_{i=1}^{h} \beta_i \le (3\kappa)/2 \bullet 2/10 = 3\kappa/10$.

For the $(3\kappa)/2$ bins with zero coarseness, suppose that there are $g \le (3\kappa)/2$ of them, each with $\sum_{i=1}^{m} W(b_i) = 1 - \beta_i$ where $\beta_i > 0$. According to the reasoning above, $\sum_{i=1}^{g} \beta_i \le (3\kappa)/2 \bullet 1/10 = 3\kappa/20$.

Therefore, $\sum_{i=1}^{l} \beta_i \le 3\kappa/10 + 3\kappa/20 = 9\kappa/20$, where $l = h + g$.

$\sum_{i=1}^{l} W(B_i) > l - 9\kappa/20$. ■

Lemma 6: *Among all the bins filled to the level of* $2/3 + 1/18 \le \sum_{i=1}^{m} b_i < 3/4$, *there are at most* κ *of them with* $\sum_{i=1}^{m} W(b_i) < 1$ *and* $m \ge 3$.

Proof: Let a bin B_i be filled with items $b_1 \ge b_2 \ge \ldots \ge b_m$ and $2/3 + 1/18 \le \sum_{i=1}^{m} b_i \le 3/4$.

If $m = 1$, then $b_1 \ge 2/3 + 1/18 > 1/2$. $\sum_{i=1}^{m} W(b_i) \ge 1$.

If $m = 2$, there are three cases to consider:

(1) If $b_1 > 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1$.

(2) If $1/3 < b_1 \le 1/2$ and $1/3 < b_2 \le 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1/2 + 1/2 = 1$.

(3) If $1/3 < b_1 \le 1/2$ and $1/6 < b_2 \le 1/3$, then $\sum_{i=1}^{m} W(b_i) \ge 6b_1/5 + 1/10 + 9(2/3 + 1/18 - b_1)/5 - 1/10 = 13/10 - 3b_1/5 \ge 1$.

Obviously, the bins with $\sum_{i=1}^{m} W(b_i) < 1$ must be assigned at least three items, i.e., $m \ge 3$. According to Lemma 1, among all bins to each of which $m \ge 3$ items are assigned, there are at most κ bins of them, each of which is no more than 3/4 full. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$. ■

Lemma 7: *Among all the bins filled to the level* $3/4 \le \sum_{i=1}^{m} b_i < 4/5$*, there are at most* κ *of them with* $\sum_{i=1}^{m} W(b_i) < 1$ *and* $m \ge 4$.

Proof: Let a bin B_i be filled with items $b_1 \ge b_2 \ge \ldots \ge b_m$ and $3/4 \le \sum_{i=1}^{m} b_i < 4/5$.

If m is equal to 1 and 2, then we can prove, similarly to the proof of Lemma 6, that $\sum_{i=1}^{m} W(b_i) \ge 1$.

If $m = 3$, there are seven cases to consider:

(1) If $b_1 > 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1$.

(2) If $1/3 < b_1 \le 1/2$ and $1/3 < b_2 \le 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1/2 + 1/2 = 1$.

(3) If $1/3 < b_1 \le 1/2$, $1/6 < b_2 \le 1/3$, and $1/6 < b_3 \le 1/3$, then $\sum_{i=1}^{m} W(b_i) = 6b_1/5 + 1/10 + 9b_2/5 - 1/10 + 9b_3/5 - 1/10 \ge 6[3/4 - (b_2 + b_3)]/5 + 9(b_2 + b_3)/5 - 1/10 = 3(b_2 + b_3)/5 + 4/5 > 1$, since $b_2 + b_3 > 1/3$.

(4) If $1/3 < b_1 \le 1/2$, $1/6 < b_2 \le 1/3$, and $b_3 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 6b_1/5 + 1/10 + 9b_2/5 - 1/10 + 6b_3/5 \ge 9b_2/5 + 6(3/4 - b_2)/5 = 3b_2/5 + 9/10 > 1$.

(5) If $1/3 < b_1 \le 1/2$ and $b_2 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 6b_1/5 + 1/10 + 6b_2/5 + 6b_3/5 = 6(\sum_{i=1}^{m} b_i)/5 + 1/10 \ge (6/5) \cdot (3/4) + 1/10 = 1$.

(6) If $1/6 < b_1 \le 1/3$, $1/6 < b_2 \le 1/3$, and $1/6 < b_3 \le 1/3$, then $\sum_{i=1}^{m} W(b_i) = 9(\sum_{i=1}^{m} b_i)/5 - 3/10 \ge (9/5) \cdot (3/4) - 3/10 > 1$.

(7) If $1/6 < b_1 \le 1/3$, $1/6 < b_2 \le 1/3$, and $b_3 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 9b_1/5 - 1/10 + 9b_2/5 - 1/10 + 6b_3/5 \ge 9(3/4 - b_3)/5 + 6b_3/5 - 2/10 > 23/20 - 3b_3/5 > 1$.

Obviously, the bins with $\sum_{i=1}^{m} W(b_i) < 1$ must be assigned at least four items, i.e., $m \ge 4$. According to Lemma 1, among all bins to each of which $m \ge 4$ items are assigned, there are at most κ bins of them, each of which is no more than 4/5 full. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$. ■

Lemma 8: *Among all the bins filled to the level* $4/5 \le \sum_{i=1}^{m} b_i < 5/6$*, there are at most* κ *of them with* $\sum_{i=1}^{m} W(b_i) < 1$ *and* $m \ge 5$.

Proof: Let a bin B_i be filled with items $b_1 \ge b_2 \ge \ldots \ge b_m$, and $4/5 \le \sum_{i=1}^{m} b_i < 5/6$.

If m is equal to 1, 2, and 3, then we can prove, similarly to the proof of Lemma 7, that $\sum_{i=1}^{m} W(b_i) \ge 1$.

If $m = 4$, there are eight cases to consider:

(1) If $b_1 > 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1$.

(2) If $1/3 < b_1 \le 1/2$ and $1/3 < b_2 \le 1/2$, then $\sum_{i=1}^{m} W(b_i) \ge 1/2 + 1/2 = 1$.

(3) If $1/3 < b_1 \le 1/2$, $1/6 < b_2 \le 1/3$, and $1/6 < b_3 \le 1/3$, then $\sum_{i=1}^{m} W(b_i) \ge 6b_1/5 + 1/10 + 9b_2/5 - 1/10 + 9b_3/5 - 1/10 \ge 6[4/5 - (b_2 + b_3)]/5 + 9(b_2 + b_3)/5 - 1/10 = 3(b_2 + b_3)/5 + 43/50 > 1$, since $b_2 + b_3 > 1/3$.

(4) If $1/3 < b_1 \le 1/2$, $1/6 < b_2 \le 1/3$, and $b_3 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) \ge 6b_1/5 + 1/10 + 9b_2/5 - 1/10 + 6(b_3 + b_4)/5 \ge 9b_2/5 + 6(4/5 - b_2)/5 = 3b_2/5 + 24/25 > 1$.

(5) If $1/3 < b_1 \le 1/2$ and $b_2 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 6b_1/5 + 1/10 + 6(b_2 + b_3 + b_4)/5 = 6(\sum_{i=1}^{m} b_i)/5 + 1/10 \ge (6/5) \bullet (4/5) + 1/10 > 1$.

(6) If $1/6 < b_1 \le 1/3$, $1/6 < b_2 \le 1/3$, $1/6 < b_3 \le 1/3$, and $1/6 < b_4 \le 1/3$, then $\sum_{i=1}^{m} W(b_i) = 9(\sum_{i=1}^{m} b_i)/5 - 4/10 \ge (9/5) \bullet (4/5) - 4/10 > 1$.

(7) If $1/6 < b_1 \le 1/3$, $1/6 < b_2 \le 1/3$ and $b_3 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 9b_1/5 - 1/10 + 9b_2/5 - 1/10 + 6(b_3 + b_4)/5 \ge 9[4/5 - (b_3 + b_4)]/5 + 6(b_3 + b_4)/5 - 2/10 > 31/25 - 3(b_3 + b_4)/5 > 1$.

(8) If $1/6 < b_1 \le 1/3$ and $b_2 \le 1/6$, then $\sum_{i=1}^{m} W(b_i) = 9b_1/5 - 1/10 + 6(b_2 + b_3 + b_4)/5 \ge 9[4/5 - (b_2 + b_3 + b_4)]/5 + 6(b_2 + b_3 + b_4)/5 - 1/10 > 67/50 - 3(b_2 + b_3 + b_4)/5 > 1$, since $b_2 + b_3 + b_4 \le 1/2$.

Obviously, the bins with $\sum_{i=1}^{m} W(b_i) < 1$ must be assigned at least five items, i.e., $m \ge 5$. According to Lemma 1, among all bins to each of which $m \ge 5$ items are assigned, there are at most κ of them, each of which is no more than 5/6 full. Therefore, there are at most κ bins with $\sum_{i=1}^{m} W(b_i) < 1$. ■

Lemma 9: *Let a bin B_i be filled with items $b_1 \ge b_2 \ge \ldots \ge b_m$. If $\sum_{i=1}^{m} b_i \ge 5/6$, then* $\sum_{i=1}^{m} W(b_i) \ge 1$.

Proof: Since $W(\beta) / \beta \ge 6/5$ in the range of $0 \le \beta \le 1/2$ and $W(\beta) = 1$ when $\beta > 1/2$, we have $\sum_{i=1}^{m} W(b_i) \ge 5/6 \bullet 6/5 = 1$. ■

Proof of Theorem 1: Suppose that in the final FF-K-packing, there are m bins $B_1, B_2, \ldots, B_m$, each of which receives at least one item, and $\sum_j W(B_j) < 1$. Let $\sum_j W(B_j) = 1 - \beta_j$, with $\beta_j > 0$ for $1 \le j \le m$.

Since our goal is to prove that $1.7L^* \ge W \ge \text{FF-K}(L) - \sum_{i=1}^{m} \beta_i$, we need to bound the quantity $\sum_{i=1}^{m} \beta_i$.

According to Lemma 8, if $\sum_{i=1}^{m} b_i \in [4/5, 5/6)$, there are at most κ bins with $m \geq 5$ and $\sum_{i=1}^{m} W(b_i) < 1$. Let l be the number of bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$ for $1 \leq l \leq \kappa$. $\sum_{i=1}^{l} \beta_i \leq \kappa(1 - 4/5 \bullet 6/5) = \kappa/25$.

According to Lemma 7, if $\sum_{i=1}^{m} b_i \in [3/4, 4/5)$, there are at most κ bins with $m \geq 4$ and $\sum_{i=1}^{m} W(b_i) < 1$. Let l be the number of bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$ for $1 \leq l \leq \kappa$. $\sum_{i=1}^{l} \beta_i \leq \kappa(1 - 3/4 \bullet 6/5) = \kappa/10$.

According to Lemma 5 and Lemma 6, if $\sum_{i=1}^{m} b_i \in [2/3, 3/4)$, then there are at most κ bins with $m \geq 3$ and $\sum_{i=1}^{m} W(b_i) < 1$. Let l be the number of bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$ for $1 \leq l \leq \kappa$. $\sum_{i=1}^{l} \beta_i \leq \kappa(1 - 2/3 \bullet 6/5) = \kappa/5$.

If $\sum_{i=1}^{m} b_i \in (2/3, 2/3 + 1/18)$, then let l be the number of bins with $m = 2$ and $\sum_{i=1}^{2} W(b_i) < 1$. Let l be the number of such bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$. According to Lemma 5, $\sum_{i=1}^{l} \beta_i \leq 9\kappa/20$.

According to Lemma 4, if $\sum_{i=1}^{m} b_i \in (1/2, 2/3]$, then there are at most κ bins with $m \geq 2$ and $\sum_{i=1}^{m} W(b_i) < 1$. Let l be the number of bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$ for $1 \leq l \leq \kappa$. $\sum_{i=1}^{l} \beta_i \leq \kappa(1 - 1/2 \bullet 6/5) = 2\kappa/5$.

According to Lemma 3, if $\sum_{i=1}^{m} b_i \in (0, 1/2]$, then there are at most κ bins with $m \geq 1$ and $\sum_{i=1}^{m} W(b_i) < 1$. Let l be the number of bins with $\sum_j W(B_j) = 1 - \beta_j$ and $\beta_j > 0$ for $1 \leq l \leq \kappa$. Then $\sum_{i=1}^{l} \beta_i \leq \kappa$.

Therefore, $\sum_{i=1}^{m} \beta_i \leq \kappa(1 + 2/5 + 9/20 + 1/5 + 1/10 + 1/25) = 2.19\kappa$.

In summary, FF-K$(L) \leq 1.7L^* + 2.19\kappa$. ■

When κ = 1, the problem becomes the well-known classical bin-packing problem. Since the ratio 1.7 is not affected by the value of κ, our result subsumes the previous known result [7]. Also, when κ = 1, the examples that can achieve the bound of 1.7 has been given in [7]. Since the term 2.19κ is a constant, it disappears when the optimal number of bins L^* approaches infinity. Therefore, we conclude that the bound is asymptotically tight.

In order to gain some insight into the average-case behavior of the new algorithm, one can analyze the performance of the algorithm under probabilistic assumptions, or conduct simulation experiments. We resort to simulation.

The simulation is conducted by running the algorithm on a large number of computer generated sample lists of items and averaging the results over a number of runs. The input data of all parameters for a list of items are generated according to uniform distribution. The number of items sharing one color is uniformly distributed in the range of $1 \le \kappa_i \le 5$. The size of an item is in the range of $0 < b_i \le 1$. The output parameter for the algorithm is the percentage of extra bins used to accommodate a list of items, with regard to the total load of the list. The total load of a list is given by $U = \sum_{i=1}^{n} \sum_{j=1}^{\kappa_i} b_{i,j}$, which is a lower bound on the number of bins needed to pack the list. In other words, the optimal number of bins needed to pack a list with a load of U is at least U. Suppose that $N(L)$ is the number of bins required by FF-K to pack a list L with a load of U, then the percentage of extra bins is given by $100 \times \frac{N(L) - U}{U}$

The result is plotted in Figure 3. The number of runs for each data point is chosen to be 20, since for our experiments, 20 runs is large enough to counter the effect of “randomness”. In order to make comparisons, we run the same data through the on-line algorithm First-Fit or First-Fit-K for $\kappa = 1$. The total load of a list is given by $\sum_{i=1}^{n} \sum_{j=1}^{\kappa_i} b_{i,j}$, which is a lower bound on the number of bins needed to pack a list of items. On the average, FF-K uses less than 10% extra bins than the best possible solution.

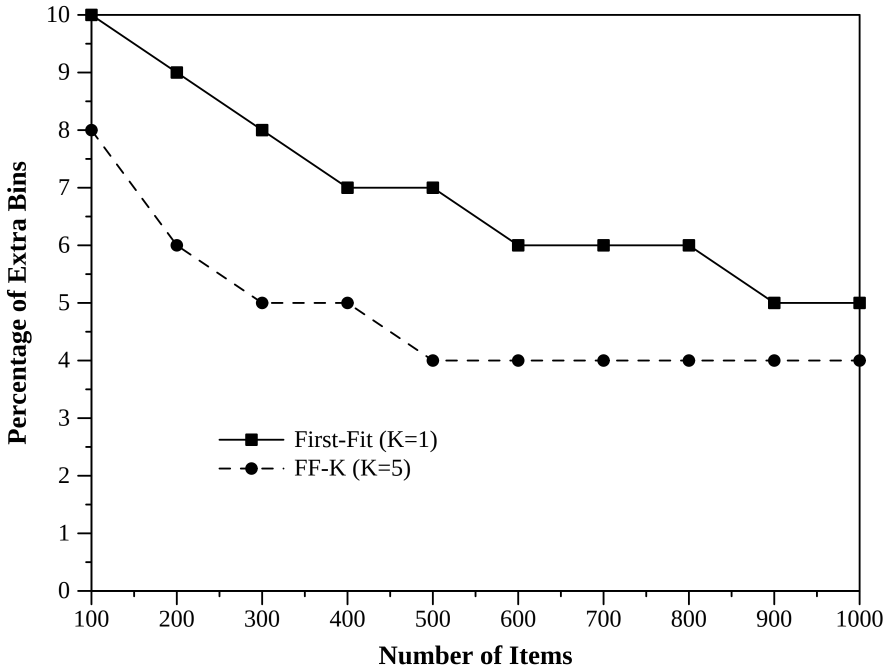

Figure 3: Performance of First-Fit-K with regard to *K*

III. Concluding Remarks

The contributions of this paper are twofold: it addresses a general problem, which is general in the sense that it occurs in different forms across various fields, and it provides a provably good solution to the general problem. The problem is motivated by a practical real-time scheduling problem and it is a generalization of the classical bin-packing problem. The heuristic algorithm First-Fit-K is shown by analysis to have an asymptotical worst-case tight bound of 1.7, and by simulation to have an average-case performance of within 10% of the optimal solution.

Much work remains to be done. One area for improvement is to lower the constant 2.19 before K. Garey et al [4] has proven an improved result, $FF(L) \le 1.7L^* + 1$, for First-Fit, where the meanings of $FF(L)$ and L^* are similarly defined. Hence we conjecture that the constant can be lowered from 2.19 to 1, if a better weighting function can be found. Also it would be interesting to conduct a probabilistic study on the performance of the proposed algorithm.

There are other heuristics that can be designed for this bin-packing problem. For example, we observe that, by numbering items with the same color from 1 to $\kappa_i \le \kappa$ and dividing the bins into κ classes and assigning the ith item of a certain color to a bin of ith class, we can ensure that items with the same colors are not assigned to the same bins. Heuristics such as First-Fit can be used to assign items to bins within each class. While the performance bounds for heuristics solving the classical bin-packing problem holds within each class, it is not clear whether it also holds for a composite algorithm which consists of the same algorithm being applied to assign items to bins in all classes.

Another direction of research will be to consider the off-line solution of the problem. We have been concerned with on-line packing so far, where the items are assigned to bins in the order they are given. It has been shown that off-line algorithm such as Next-Fit Decreasing and First-Fit Decreasing generally delivers better performance than on-line counterparts. It would be interesting to derive the performance bounds for these simple algorithms. These are the problems that we are currently studying.

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