Edge-Labelled Gossip in Rounds

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Edge-Labelled Gossip in Rounds

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Abstract

In this paper, we consider a gossiping problem in which individuals in a network exchange information periodically according to a fixed schedule. In particular, a proper edge-coloring can be considered as a schedule for the calls of a gossiping scheme. We have investigated the time required to complete gossiping in graphs under various colorings. In particular, we have determined the minimum time to complete gossiping under any proper edge-coloring of a path and given bounds on the time required to complete gossiping in cycles and in trees with bounded degree.

Key Words: gossiping, graphs, networks, edge-coloring.

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1. Definitions

Gossiping is the process in which each member of a set of individuals knows a unique piece of information initially and must learn all of the information of the group through some set of communications (calls). This process is modeled as a problem on graphs in which the vertices represent individuals and edges represent allowed communication links. Several variations of this problem have been considered (see [2]). We are interested in a particular situation in which specific calls are made periodically according to a predetermined schedule. In practical terms, such a schedule allows efficient communication while avoiding the possibility of message collisions or congestion.

Let G be any connected graph on n vertices and let C be a proper k edge-coloring of G which assigns the colors $1, 2, \ldots, k$ to the edges of G such that no vertex is incident on more than one edge of any color i. (Henceforth, all edge-colorings will be assumed to be proper and we will omit the word.) If all k colors are used in the edge-coloring, we say that it is a **strict** k **edge-coloring**. At time 0, every vertex has a unique piece of information known only to itself. At each time t, $t = 1, 2, 3, \ldots$, each pair of vertices which are joined by an edge colored $i \equiv t (mod\, k)$ exchange all of the information they know at that time. Note that operations on colors are performed modulo k, adopting the convention that color k is used in place of color 0. Let t_C denote the earliest time t at which every vertex has learned all n pieces of information given the edge-coloring C. We say that G **completes gossiping** under edge-coloring C at time t_C. The k **gossip time** of G, denoted $\bar{g}_k(G)$, is defined to be the minimum t_C over all k edge-colorings C of G. The **strict** k **gossip time** of G, denoted $g_k(G)$, is defined to be the minimum t_C over all strict k edge-colorings C of G. It follows from the definitions that $g_k(G) \geq \bar{g}_k(G)$.

2. Gossiping on Paths

Let $P_n = (v_0, v_1, \ldots, v_n)$ denote the path consisting of n edges and $n+1$ vertices. It is convenient to use an n character k-ary string $c_1 c_2 \ldots c_n$, where $c_i \in \{1, 2, \ldots, k\}$ for all i, to denote a particular k edge-coloring of P_n. In particular, a (proper) k edge-coloring corresponds to such a string in which $c_i \neq c_{i+1}$ for $1 \leq i < n$. A strict k edge-coloring has the additional property that for each $j \in \{1, \ldots, k\}$ there is at least one i, $1 \leq i < n$, such that $c_i = j$.

The following simple observations, stated without proof, will be useful below.

1. For a given k edge-coloring C of P_n, let t_{0n} denote the time required for a message to travel from v_0 to v_n and let t_{n0} denote the time required for a message to travel from v_n to

v_0. The k gossip time of P_n under edge-coloring C is the maximum of these two times, that is, $t_C = \max\{t_{0n}, t_{n0}\}$.

2. The k edge-coloring $c_1 c_2 \ldots c_n$ and its reverse, $c_n c_{n-1} \ldots c_1$, complete gossiping at the same time.

Lemma 1: Given any k edge-coloring C of P_n with $c_1 > 1$, there is another k edge-coloring C' of P_n such that $c'_1 = 1$ and $t_{C'} \le t_C$.

Proof: Since the k edge-coloring C and its reverse C' complete gossiping at the same time, if $c_n = 1$ then $C' = c_n c_{n-1} \ldots c_1$ is such a k edge-coloring.

We can now restrict our attention to k edge-colorings C of P_n with $c_1 > 1$ and $c_n > 1$. Without loss of generality, we can assume that $c_1 < c_n$. From such a C, we construct a new k edge-coloring $C' = c'_1 c'_2 \ldots c'_n$ by shifting the numbers corresponding to the colors by c_1 modulo k. For each i, let $c'_i = (c_i - c_1 + 1) \pmod k$, recalling that 0 is represented by color k. In particular, this means that $c'_1 = 1$. Messages traveling through any interior node of P_n encounter the same delays with either coloring since the difference between two consecutive labels remains the same modulo k. Furthermore, the endpoints send their own messages $c'_1 - 1$ time units earlier than under the previous edge-coloring. Thus, in this case $t_{C'} \le t_C$. □

Note that the edge-coloring constructed in the proof of Lemma 1 is strict if and only if the initial edge-coloring is strict.

Corollary 2: There is a strict k edge-coloring C of P_n with $c_1 = 1$ which completes gossip in time $g_k(P_n)$ and a k edge-coloring C' of P_n with $c'_1 = 1$ which completes gossip in time $\bar{g}_k(P_n)$.

Proof: This follows immediately from Lemma 1. □

Thus, in searching for (strict) k edge-colorings of paths that complete gossiping in the minimum time, we need only consider those in which the first edge is colored 1.

Theorem 3: $$g_k(P_n) \ge \bar{g}_k(P_n) \ge \begin{cases} \frac{k(n-1)}{2} + 1, & n \text{ odd}, n \ge k \ge 2 \\ \frac{kn}{2} + 1, & n \text{ even}, n \ge k \ge 2. \end{cases}$$

Proof: The first inequality follows from the definitions of $g_k(G)$ and $\bar{g}_k(G)$. Note that the requirement that $n \ge k$ is due to the definition of strict k edge-coloring. The same bound actually holds for $\bar{g}_k(P_n)$ for any $n \ge 2$.

We begin by considering the simple case $k=2$. Clearly, $\bar{g}_2(P_n) \geq n$ (and hence $g_2(P_n) \geq n$) since the information can travel over at most one edge per time unit and the two endpoints v_0 and v_n are at distance n from each other. For even n, the only 2 edge-coloring with $c_1 = 1$, 12121...212, is strict and has $c_n = 2$. The information from v_n leaves v_n at time 2 and is thereby delayed by one time unit. Thus, $\bar{g}_2(P_n) \geq n+1$ (and $g_2(P_n) \geq n+1$) for even n. (In fact, we have shown equality for even n.)

Next, we consider $k=3$, that is, we show that $\bar{g}_3(P_n) \geq \frac{3n-1}{2}$ for odd $n \geq 2$ and that $\bar{g}_3(P_n) \geq \frac{3n}{2}+1$ for even $n \geq 2$. We will then generalize this argument for larger k.

In any 3 edge-coloring of P_n with adjacent edges (u,v) and (v,w) colored i and j, respectively, we can assume that $j = i+1$, without loss of generality. Any information traveling in the direction from u to w is forwarded by v immediately so that the information is known to w exactly 2 time units after it is sent by u. However, any information traveling from w to u is delayed for one time unit at v and it is known to u exactly 3 time units after it is sent by w. Thus, any intermediate vertex in the path P_n adds one unit of delay to a message traveling through it in one direction and no delay to a message traveling through it in the opposite direction.

In order to complete gossiping in a given edge-coloring of P_n, the two endpoints must each learn each other's information. In particular, this means that one message must travel from v_0 to v_n and another must travel from v_n to v_0. The time required for either of these messages to arrive at is destination is a lower bound on the time required to complete gossiping. Each of these messages passes over n edges and encounters $n-1$ intermediate vertices. Each message requires exactly one unit of time to travel over each edge and each intermediate vertex delays exactly one of the messages by one time unit. Thus, the sum of the times taken by these two messages is $3n-1$ and $\bar{g}_3(P_n) \geq \lceil \frac{3n-1}{2} \rceil$.

We can improve this lower bound slightly for the case of even n. In particular, let $n = 2p$ while $k=3$. Let us assume, by way of contradiction, that gossip can be completed in P_n under some 3 edge-coloring C in time $\lceil \frac{3n-1}{2} \rceil = 3p$. As above we will restrict our attention to the messages sent between the endpoints. Since we can also assume that $c_1 = 1$, the endpoint v_0 must learn v_n's information at some time $t \equiv 1 \pmod 3$. Since $t \leq 3p$, this must occur no later than at time $3p-2 \equiv 1 \pmod 3$. Thus, the message from v_n to v_0 encountered at most $3p-2-n = p-2$ delays at intermediate vertices. Since the total number of delays incurred by the two messages sent between the endpoints is $n-1$, the message from v_0 to v_n must encounter at least $(n-1)-(p-2)=p+1$ delays and can arrive at v_n no sooner than time $n+p+1=3p+1$, contradicting the assumption.

This argument is easily generalized for arbitrary k. In any k edge-coloring of P_n with adjacent edges (u,v) and (v,w) colored i and j, respectively, we can assume that $j>i$, without loss of generality. Any information traveling from u to w is delayed $j-i-1$ time units at vertex v whereas any information traveling from w to u is delayed $k-(j-i)-1$ time units at vertex v. Each of the messages traveling between v_0 and v_n passes over n edges and encounters $n-1$ intermediate vertices. Each message requires exactly one unit of time to travel over each edge while each intermediate vertex introduces a total of $k-2$ units of delay to the two messages. Thus, the sum of the times taken by these two messages is $2n+(n-1)(k-2)=(n-1)k+2$ and $\bar{g}_k(P_n)\geq\lceil\frac{(n-1)k+2}{2}\rceil$.

As before, we can improve this bound slightly for the case of even n. In particular, let $n=2p$. We assume, by way of contradiction, that gossip can be completed in P_n under some k edge-coloring C in time $\frac{n}{2}k$. Since $c_1=1$, v_0 must learn v_n's information at some time $t\leq\frac{n}{2}k=pk$ where $t\equiv 1 \pmod{k}$. So, in fact, v_0 must learn v_n's information at some time $t\leq(p-1)k+1$. Since the path contains n edges, each of which requires one unit of time to cross, the message from v_n to v_0 incurred at most $(p-1)k+1-n=p(k-2)-k+1$ delays at the intermediate vertices. Since a total of $(n-1)(k-2)$ delays are incurred by the two messages sent between the endpoints, the message from v_0 to v_n must encounter at least $(n-1)(k-2)-(p(k-2)-k+1)=p(k-2)+1$ delays and can arrive at v_n no sooner than time $p(k-2)+1+2p=pk+1$. □

Before stating general upper bounds for $g_k(P_n)$ and $\bar{g}_k(P_n)$, we consider the special case $n=k$ under strict edge-coloring.

Lemma 4: $$g_k(P_k)=\begin{cases}\frac{k(k-1)}{2}+2, & \text{odd } k\geq 3\\ \frac{k^2}{2}+1, & \text{even } k\geq 4.\end{cases}$$

Proof: For odd $k\geq 3$, the strict edge-coloring $135\ldots k(k-1)(k-3)\ldots 2$ has $t_{0n}=\frac{k(k-1)}{2}+2$ and $t_{n0}=\frac{k(k-1)}{2}+1$. In this case, the lower bound from Theorem 3 can be improved. The edge-coloring must be a permutation of $1,2,\ldots,k$. In particular, if we assume that the edge from v_1 to v_2 is colored 1, then the edge from v_k to v_{k-1} must be colored i for some $i>1$. This means that a delay of $i-1\geq 1$ time units is introduced at v_k. The sum of the delays incurred by the two messages sent between the endpoints is at least $k(k-1)+1$. Since every edge must be traversed by each of these two messages, the sum of the times taken by the two messages is $2k+(k-2)(k-1)+1$, so $g_k(P_k) \geq \lceil \frac{2k+(k-2)(k-1)+1}{2} \rceil = \frac{k(k-1)}{2}+2$.

For even $k \geq 4$, the strict edge-coloring $135\ldots(k-1)k(k-2)(k-4)\ldots2$ has $t_{0n} = \frac{k^2}{2}-k+2$ and $t_{n0} = \frac{k^2}{2}+1$. The lower bound comes directly from Theorem 3. □

Theorem 5: $\bar{g}_k(P_n) = \begin{cases} \frac{k(n-1)}{2}+1, & n \text{ odd}, n \geq k \geq 2 \\ \frac{kn}{2}+1, & n \text{ even}, n \geq k \geq 2. \end{cases}$

Proof: For odd n, the non-strict 2 edge-coloring $12121\ldots21$ has $t_{0n} = t_{n0} = \frac{k(n-1)}{2}+1$. For even n, the non-strict edge-coloring 1212...12 has $t_{0n} = \frac{kn}{2}$ and $t_{n0} = \frac{kn}{2}+1$. The matching lower bounds come directly from Theorem 3. □

Theorem 6: $g_k(P_n) = \begin{cases} \frac{k(n-1)}{2}+1, & n \text{ odd}, n > k \geq 2 \\ \frac{kn}{2}+1, & n \text{ even}, n \geq k \geq 2. \end{cases}$

Proof: We will prove the following claim from which the result follows. "There exist colorings with the following properties: For even n, $c_n = 2$, $t_{0n} = \frac{kn}{2}-k+2$ and $t_{n0} = \frac{kn}{2}+1$. For odd n, $c_n = 1$ and $t_{0n} = t_{n0} = \frac{k(n-1)}{2}+1$.

We prove this claim by induction. Regardless of the parity of k, the basis of the induction is for even n. When k is even, the basis is given in the proof of Lemma 4. When k is odd, the sequence $123\ldots(\frac{k+1}{2})k(k-1)\ldots(\frac{k+1}{2}+1)2$ with $n = k+1$ provides the basis.

Note that if a color sequence ends with a 2, then appending a 1 causes the leftward time (t_{0n}) to increase by $k-1$ while the rightward time (t_{n0}) stays the same. Further, if a color sequence ends with a 1, then appending a 2 causes the leftward time to increase by 1 and the rightward time to increase by k.

Using these observations it is easy to show that if the claim is true for an even (odd) n then it follows for the next odd (even) $n+1$. In the even case a 2 is appended and in the odd case a 1 is appended. For example, suppose the leftward time for an odd n is $\frac{k(n-1)}{2}+1$ and the final color is 1. If a 2 is appended, the leftward time for the even $n+1$ is $(\frac{k(n-1)}{2}+1)+1 = \frac{k(n+1)}{2}-k+2$.

The lower bounds come directly from Theorem 3. □

Lemma 7: Given any (strict) k edge-coloring C of P_n, $t_C \leq k+(k-1)(n-1)$.

Proof: We construct a specific coloring strict k edge-coloring C of P_n which maximizes the time required for a message to travel from v_0 to v_n. In order to delay the starting time of the message from v_0 as long as possible, let $c_1 = k$. For any intermediate vertex v_i, the message can be delayed by at most $k-1$ time units. By setting c_{i+1} to $c_i+(k-1)(\text{mod } k)$ for each i, the message is delayed as much as possible and arrives at v_n at time $k+(k-1)(n-1)$. Thus, $t_C = k+(k-1)(n-1)$. It is clear from the construction that no other k edge-coloring of P_n could require more time to complete gossiping. □

3. Gossiping in Trees of Bounded Degree

Let T denote a tree on n vertices. We use d_T to denote the diameter of T and Δ_T to denote the maximum degree of any vertex in T. The subscripts may be omitted when the meaning is clear from context. Clearly, any proper edge-coloring of T must use at least Δ_T colors.

Lemma 8: Given a tree T of diameter d and maximum degree Δ, $g_k(T) \geq \bar{g}_k(T) \geq g_k(P_d)$ for any $k \geq \Delta$.

Proof: The first inequality follows from the definitions of g_k and $\bar{g}_k$. The second follows from the observation that information must travel between two vertices which are at a distance Δ apart and the fact that between any such pair of vertices there is a unique path in T. □

Let us consider the special case of trees T where $\Delta_T = 3$. For a fixed diameter d, consider the largest (most vertices) tree with diameter d and maximum degree 3 which we will call T_d. T_d has $1+3\sum_{r=1}^{d/2} 2^{r-1}$ vertices if d is even and $2\sum_{r=1}^{(d+1)/2} 2^{r-1}$ vertices if d is odd (see [1]). We can draw T_d (as an example, T_7 is shown in Figure 3-1) as two full regular binary trees T_a and T_b rooted at adjacent vertices a and b, respectively, where the height of T_a is $\lceil \frac{d}{2} \rceil$ and the height of T_b is $\lfloor \frac{d}{2} \rfloor$.

Since every tree T with diameter d and maximum degree 3 is a subtree of T_d, any proper edge-coloring of T_d (when restricted to the edges of T) is a proper edge-coloring of T. Thus, any upper bound on the gossip time of T_d is an upper bound on the gossip time of T.

Theorem 9: If T is a tree with diameter d and maximum degree 3, then $\bar{g}_3(T) \leq g_3(T) \leq 2d+1$.

Proof: Consider any proper 3 edge-coloring of T_d. The length of the longest path between any two leaves of T_d, say u and v, is d. Consider the edge labels on the path from u to v. The information must

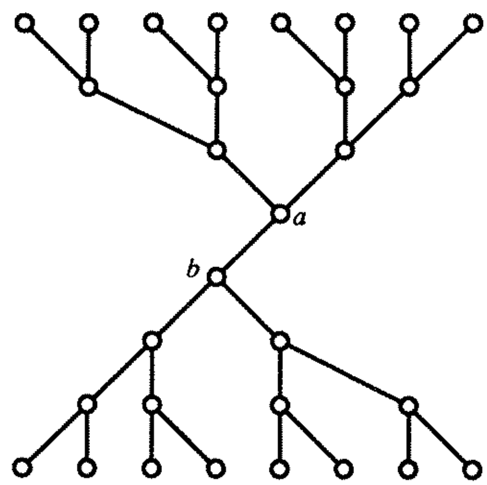

Figure 3-1: largest tree with diameter 7 and maximum degree 3

leave u no later than at time 3 and then travel over $d-1$ other edges before arriving at v. In addition to the time taken to travel over these edges, the message may be delayed by no more than 1 time unit at any of the $d-1$ intermediate vertices. Thus, it must arrive at v no later than time $3+(d-1)+(d-1)=2d+1$, so $g_3(T_d) \leq 2d+1$. The same edge-coloring restricted to the edges of T yields a coloring of T which completes gossiping at time no later than $2d+1$, so $g_3(T) \leq 2d+1$. □

An improved lower bound can been shown for the tree T_d.

Theorem 10: $2d-1 \leq \bar{g}_3(T_d) \leq g_3(T_d)$.

Proof: Consider all of the messages travelling from a bottom leaf to a top leaf of the tree T_d (where bottom and top are as depicted in Figure 3-1). Let c be an internal node in the lower subtree rooted at b. All messages coming up to c either from its left subtree or from its right subtree must be delayed for one unit at c. Let d be an internal node in the upper subtree rooted at a. Messages coming up to any such node d and bound for leaves in the upper subtree rooted at d may be delayed at d. In particular, all such messages destined for nodes in either the left upper subtree or the right upper subtree rooted at d will be delayed for one unit at node d. Hence, at least one message from a bottom leaf to a top leaf will encounter delays at every intermediate step. The total time taken by this message (if there is no initial delay) is $d+(d-1)=2d-1$. □

The proof of Theorem 9 is easily generalized to produce an upper bound on the gossip time in trees with diameter d and maximum degree Δ.

Theorem 11: If T is a tree of diameter d and maximum degree Δ, then

$\bar{g}_\Delta(T) \le g_\Delta(T) \le (\Delta-1)d+1$.

Proof: Consider any proper Δ edge-coloring of $T_{\Delta,d}$, the largest tree with diameter d and maximum degree Δ. The length of the longest path between any two leaves of $T_{\Delta,d}$, say u and v, is d. Consider the edge labels on the path from u to v. The information must leave u no later than at time Δ and then travel over $d-1$ other edges before arriving at v. In addition to the time taken to travel over these edges, the message may be delayed by no more than $\Delta-2$ time units at any of the $d-1$ intermediate vertices. Thus, it must arrive at v no later than time $\Delta+(d-1)+(\Delta-2)(d-1)=(\Delta-1)d+1$, so $g_\Delta(T_{\Delta,d}) \le (\Delta-1)d+1$. The same edge-coloring restricted to the edges of T yields a coloring of T which completes gossiping at time no later than $(\Delta-1)d+1$, so $g_3(T) \le (\Delta-1)d+1$. □

4. Gossiping in Cycles

Let C_n denote the cycle on n vertices labelled $v_1, v_2, \ldots, v_n$. We can use an n character k-ary string $c_1 c_2 \ldots c_n$ to denote a particular k edge-coloring of C_n. In this case, notice that for the edge-coloring to be proper, it is necessary that $c_1 \ne c_n$. As with paths, any edge-coloring and its reverse complete gossiping at the same time. Furthermore, any edge-coloring $c_1 c_2 \ldots c_n$ and its circular shifts $c_2 \ldots c_n c_1$, $c_3 \ldots c_n c_1 c_2$, ... and $c_n c_1 \ldots c_{n-1}$ all complete gossiping at the same time.

Lemma 12: Given any k edge-coloring C of C_n with $c_1 > 1$, there is another k edge-coloring C' of C_n such that $c'_1 = 1$ and $t_{C'} \le t_C$.

Proof: Since the k edge-coloring C and its circular shifts all complete gossiping at the same time, if $c_i = 1$ then $C' = c_i c_{i+1} \ldots c_n c_1 \ldots c_{i-1}$ is such a k edge-coloring.

We can now restrict our attention to k edge-colorings C of P_n with $c_i > 1$ for all $1 \le i \le n$. Choose i such that $c_i \le c_j$ for all $j \ne i$. Consider the edge-coloring $C' = 1\, c_{i+1} \ldots c_n c_1 \ldots c_{i-1}$, that is, the k edge-coloring obtained from C by shifting c_i into the first position and replacing it with 1. It is easy to see that the time to complete gossip in C' is no more than that in C. By our choice of i, we know that the neighboring edges had larger labels in C. Any message arrives at an endpoint of the relabelled edge at the same time in C' as in C. The message is transmitted over the relabelled edge exactly c_i-1 time units earlier than in C. The message is then forwarded from the other endpoint at the same time as in C. Thus, $t_{C'} \le t_C$. □

Note that if the initial edge-coloring is strict, then the edge-coloring constructed in the proof of Lemma 12 is a circular shift of the original and thus also strict.

Corollary 13: There is a strict k edge-coloring C of C_n with $c_1 = 1$ which completes gossip in time $g_k(C_n)$ and a k edge-coloring C' of C_n with $c'_1 = 1$ which completes gossip in time $\bar{g}_k(C_n)$.

Proof: This follows immediately from Lemma 12. □

Thus, in searching for (strict) k edge-colorings of cycles that complete gossiping in the minimum time, we need only consider those in which the first edge is colored 1.

Theorem 14: $\bar{g}_k(C_n) \geq \frac{(k-1)n}{k} + \frac{4}{k} - 2$ and $g_k(C_n) \geq \frac{(k-1)n}{k} + \frac{4}{k} - 2 + \frac{(k-1)(k-2)}{kn}$.

Proof: Under any edge-coloring C, the information originating at vertex a must arrive at distinct and adjacent vertices b (following a path counterclockwise from a) and c (following a path clockwise from a) in time $\leq t_C$. Let l_{ab} be the number of edges in the (counterclockwise) path from a to b and l_{ac} be the number of edges in the (clockwise) path from a to c. Note that $l_{ab}+l_{ac} = n-1$. Similarly, let d_{ab} and d_{ac} be the number of delays encountered by the messages on these paths. It follows that $t_C \geq l_{ab}+d_{ab}$ and $t_C \geq l_{ac}+d_{ac}$.

Suppose the Theorem is false, i.e., under some edge-coloring C gossip completes in time $t_C < \frac{(k-1)n}{k} + x$, where x depends on whether or not strict edge-coloring is required. Then, for every a, $l_{ab} > \frac{n}{k} - x - 1$, otherwise it must be the case that $l_{ac} \geq \frac{(k-1)n}{k} + x$. Similarly, we can argue that $l_{ac} > \frac{n}{k} - x - 1$. If $S = \Sigma_a\ (l_{ab}+d_{ab}+l_{ac}+d_{ac})$ then $\frac{S}{2n} < \frac{(k-1)n}{k} + x$ must also hold. We show that $\frac{S}{2n} \geq \frac{(k-1)n}{k} + x$.

Let $S = S_l + S_d$, where $S_l = \Sigma_a\ (l_{ab}+ l_{ac}) = n(n-1)$ and $S_d = \Sigma_a\ (d_{ab}+ d_{ac})$. Consider a fixed vertex p as a varies. Recall that at any p, the sum of the delay incurred by a message traveling clockwise and the delay incurred by a message traveling counterclockwise is $k-2$. For $\frac{n}{k} - x - 2$ choices of a the counterclockwise delay of p must contribute to d_{ab} and for $\frac{n}{k} - x - 2$ choices of a the clockwise delay of p must contribute to d_{ac}. Hence $S_d \geq (k-2)n(\frac{n}{k} - x - 2) + S_d'$ where S_d' is the sum of the delays encountered at the beginning for each a that are not counted in the above argument.

For the non-strict case, we know that at each a one of the incident edges is > 1, so there is a delay of at least 1 in that direction. Hence, $S_d' \geq n$. For the strict case, we know that there are choices of a which must have incident edges labelled with 2, 3, ..., k. It follows that $S_d' \geq (n-k) + k(k-1)$.

Using the above inequalities it follow that, in both cases, $\frac{S}{2n} \geq \frac{(k-1)n}{k} + x$. □

Lemma 15: For even $n \geq 4$, $g_2(C_n) = \bar{g}_2(C_n) = \frac{n}{2}$.

Proof: Since the diameter of C_n is $\frac{n}{2}$, we know that $g_2(C_n) \geq \frac{n}{2}$ and $\bar{g}_2(C_n) \geq \frac{n}{2}$ from Theorem 14. The strict 2 edge-coloring 1212...12 completes gossiping in $\frac{n}{2}$ time units.

Lemma 16: For $n \geq 4$, $g_3(C_n) \leq \lfloor \frac{3n+1}{4} \rfloor$.

Proof: The edge-colorings 1212...123 and 1212...1213 for odd and even n, respectively, complete gossip in $\lfloor \frac{3n+1}{4} \rfloor$ time units. □

Lemma 17: For $n \geq 4$, $\bar{g}_3(C_n) \leq \lceil \frac{n}{2} \rceil + \lceil \frac{n-4}{4} \rceil$.

Proof: The edge-colorings 1212...123 and 1212...12 for odd and even n, respectively, complete gossip in $\lceil \frac{n}{2} \rceil + \lceil \frac{n-4}{4} \rceil$ time units. □

Theorem 18: For $n = ik + j$ such that $1 \leq j \leq k$, $\bar{g}_k(C_n) \leq g_k(C_n) < \frac{(k-1)n}{k} + \frac{(2k-j-1)(k-1)}{k}$.

Proof: For $n = ik+j$ where $2 \leq j \leq k$, consider the strict edge-coloring $123\ldots k\ldots 123\ldots k123\ldots j$, that is, c_i is set to be $i \bmod k$. Under this particular edge-coloring, information travels much more quickly in one direction (say clockwise) than in the opposite direction (counter-clockwise). The information beginning at any vertex must be received by all the other vertices before gossip is completed. The last vertices to send their information are those vertices v_i where $i \equiv 0 \bmod k$, that is, those vertices incident on edges with labels $k-1$ and k. Such a vertex, say u, begins by sending its information counter-clockwise at time $k-1$ and then sends its message clockwise at time k. The number of vertices informed by time t along a counter-clockwise path from u is roughly $\lfloor \frac{t}{k-1} \rfloor$ for $t \geq k-1$. When proceeding counter-clockwise from v_1, $c_1 = 1$ is followed by $c_n = j$. For some starting vertices, this pair of edges will be traversed counter-clockwise during the gossiping scheme, informing vertices more quickly. Thus, the number of vertices informed by time t along a counter-clockwise path from u is $\geq \lfloor \frac{t}{k-1} \rfloor$ for $t \geq k-1$. In the clockwise direction, $c_n = j$ is followed by $c_1 = 1$, so a message travelling over this pair of edges is delayed by $k-j$ time units at v_1. The number of vertices informed by time t along a clockwise path from u is $\geq t-(k-1)-(k-j)$ for $t \geq k-1$. Thus, all n vertices will know u's information and gossip will be completed at time no later than t such that $n = 1 + (t-(k-1)-(k-j)) + \lfloor \frac{t}{k-1} \rfloor$. Solving for t we get, $t < \frac{(k-1)n}{k} + \frac{(2k-j-1)(k-1)}{k}$.

When $n = ik+1$, the above edge-coloring is not proper. In this case, we consider the strict edge-coloring $123\ldots k\ldots 123\ldots k2$, that is, c_i is set to be $i \bmod k$ for $i < n$ and $c_n = 2$ and can show that $t < \frac{(k-1)n}{k} + \frac{2(k-1)(k-1)}{k}$. □

5. Remarks

The following simple bounds on the gossip time of an arbitrary connected graph on n vertices follow from the definition of diameter and Lemma 7.

Lemma 19: Given any n vertex connected graph G with diameter d, $d \le \overline{g}_k(G) \le k + (k-1)(d-1)$ and $d \le g_k(G) \le k + (k-1)(d-1)$.

Except for these trivial bounds, the problem is open for arbitrary connected graphs.

Similar bounds for arbitrary trees may be derived from the definition of diameter and Theorem 11. Otherwise, the problem is open for arbitrary trees.

We believe that the upper bounds of Lemmas 16 and 17 are, in fact, equalities. In particular, equality holds for $4 \le n \le 16$.

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