An Algorithm for State Constrained Stochastic Linear-Quadratic Control

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***Abstract*—Here we consider a state-constrained stochastic linear quadratic control problem. This problem has linear dynamics and a quadratic cost, and states are required to satisfy a probabilistic constraint. In this paper, the joint probabilistic constraint in the model is converted to a conservative deterministic one using multi-dimensional Chebyshev bound. A maximum volume inscribed ellipsoid problem is solved to obtain this probability bound. We then design an optimal affine controller for the resulting problem. The convexity of the Chebyshev bound-constrained problem is proved and a practical algorithm is developed. Two numerical examples show that the algorithm is very reliable even when the disturbances are big and the problem horizon grows to as long as 20 stages. It is also shown that the approach proposed in this paper can be used to reformulate some classical problems such as tracking problems.**

I. Introduction

In this paper we consider a probability constrained discrete-time stochastic linear-quadratic control problem with additive, zero mean and finite second moment disturbances. This problem is repeatedly solved in stochastic model predictive control scheme. Unconstrained discrete-time stochastic LQ control has been extensively studied over the last half-century, and it is well-known that there exists a closed form optimal solution which can be expressed in terms of the discrete-time algebraic Riccati equations, provided the system is controllable and observable (see [1] and references therein). However, constrained problems present unique challenges that are generally not addressed by classical methods.

Recently, probability constrained linear quadratic control has been a very active research area in the control community (e.g., see [3], [4], [2] and [5]), as it is a natural extension of the constrained deterministic linear quadratic control. Probability constrained optimization problems were first studied by Charnes, Cooper, and Symonds [6], Miller and Wagner [7] and Prekopa [8]. Hard, robust constraints (see [9]) can be viewed as probability constraints that must hold with probability 1. Interestingly, sometimes a small relaxation of this probability requirement can lead to a significant improvement in the achievable objective function value. Currently there are two main-stream approaches dealing with probability constraints: probabilistic approximation (see [15], [11] and [10]) and sampling (see [12] and [13]). However, existing results primarily deal with scalar cases. To the authors' knowledge, no practical methods exist for multi-dimensional probability constraints.

In deterministic LQR problems, open-loop and closed loop strategies are equivalent. So, in finite horizon problems, an optimal control sequence can be computed by solving a constrained quadratic program, even when inequality constraints on states and controls are present. In a stochastic setting, performance can be improved by utilizing a closed loop controller. That is, knowledge of past disturbances can be incorporated into the control actions taken at each time period. Here we consider the design of closed loop controllers that minimize an LQR cost subject to a probability constraint on the system's states.

The main purpose of this paper is to derive a tractable approximation of the state-constrained stochastic LQR problem, and develop an algorithm for computing an optimal affine controller for this approximation. We summarize an algorithm that is practical and can be implemented with conventional optimization solvers. Moreover, the controllers produced by this algorithm are guaranteed to be feasible with respect to the probability constraints in the state-constrained stochastic LQR problem.

The rest of the paper is organized as follows: In section II, we formulate the state-constrained stochastic LQR model. In section III, we propose an inner approximation of the probabilistic state constraints using a multi-dimensional Chebyshev bound. In section IV, we specify a convex program that can be solved to compute an optimal causal, affine controller for the approximated state-constrained stochastic LQR problem. We also summarize the algorithm framework for the approximate problem. In section V, two numerical examples are shown to demonstrate the reliability of the approach, by comparing our approach with the certainty equivalent model. Finally, conclusions and a discussion of future work are given in section VI.

II. Problem formulation

The probabilistic state constrained stochastic LQR model is formulated as:

$$\text{minimize:} \quad E[\textstyle\sum_{k=0}^{N-1}(x_k^TQx_k+u_k^TRu_k)+x_N^TQ_Nx_N]$$

$$\text{subject to:} \quad x_{k+1}=Ax_k+Bu_k+w_k \quad \text{for } k=0,\dots,N-1$$

$$\mathbf{P}\left(T_1x_1+\cdots+T_Nx_N\le b\right)\ge\alpha$$

We call this problem **(P1)**. In this formulation, we assume the initial state vector x_0 is known. N is the problem horizon, $x_k\in R^n$ is the system state at time k, $u_k\in R^n$ the control applies at time k and $w_k\in R^n$ is the random disturbance applied to the system at time k. The w_k are stage-wise independent with zero mean and covariance matrices Σ_k. Q_N is a terminal cost matrix. The constraint is a probabilistic constraint on the states over the entire problem horizon.

In this paper, we aim to develop an algorithm for computing closed-loop controllers. Rather than solve this problem exactly, we will replace the probability constraint with an inner approximation. A controller that is feasible for this approximation will be guaranteed to be feasible for the original probability constraint. Our goal is to find a reliable and fast way to solve this problem. First we rewrite the system dynamics in matrix form. Since the system dynamics are linear, after some algebra, we get:

$$\mathcal{X} = \mathbb{F}x_0 + \mathbb{H}\mathcal{U} + \mathbb{G}\mathcal{W}$$

where

$$\mathcal{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \mathcal{U} = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}, \mathcal{W} = \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix}$$

The block matrices $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{G}$ (note that for convenience the indices start with 0) are given by:

$$\mathbb{F} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix}, \ \mathbb{H} = \begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & & \ddots & \\ h_{N-1} & h_{N-2} & \dots & h_0 \end{bmatrix}$$

$$\mathbb{G} = \begin{bmatrix} g_0 & & & \\ g_1 & g_0 & & \\ \vdots & & \ddots & \\ g_{N-1} & g_{N-2} & \cdots & g_0 \end{bmatrix}$$

$$f_k = A^{k+1}, \ h_k = A^k B, \text{ and } g_k = A^k.$$

$$k = 0, 1, \dots, N-1.$$

Observse that $\mathcal{W}$ has zero mean and its block covariance matrix Σ is given by:

$$\Sigma = \begin{bmatrix} \Sigma_0 & & \\ & \ddots & \\ & & \Sigma_{N-1} \end{bmatrix}$$

Our next step is to reformulate **(P1)** with this new notation. Let the diagonal $N \times N$ block matrices $\mathbb{Q}$ and $\mathbb{R}$ be defined as follows:

$$\mathbb{Q} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & Q_N \end{bmatrix}, \ \mathbb{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$

Using the notation above, we can rewrite **(P1)** as a more compact form.

$$\begin{aligned} \text{minimize:} \quad & E[\mathcal{X}^T\mathbb{Q}\mathcal{X} + \mathcal{U}^T\mathbb{R}\mathcal{U}] + x_0^T Q x_0 \\ \text{subject to:} \quad & \mathbf{P}\left(T\mathcal{X} \le b\right) \ge \alpha \end{aligned} \tag{1}$$

where T is a matrix constructed by concatenating the matrices $T_1, \dots, T_N$. In the next section, we will replace the probabilistic state constraint in (1) by a multi-dimensional Chebyshev inequality.

III. Approximation of the probabilistic constraint

In this section we present a conservative approximation for the probabilistic constraint in (1), using the multi-dimensional Chebyshev inequality. Before we derive the main result of this paper, we mention that there exist several approaches to tackle scalar chance constraints, namely Bernstein approximations and scenario approximation. The goal for this section is to derive a method to approximate multidimensional chance constraints, which serves as the foundation of our control algorithm in the later sections. The following theorem provides a multi-dimensional Chebyshev inequality [14]:

Theorem 1. *Let z be a random vector in $\mathbb{R}^d$ and $\mathcal{S}$ a subset of $\mathbb{R}^d$ defined by a collection of linear inequalities. If $P \in \mathbb{S}^d$, $q \in \mathbb{R}^d$ and $r \in \mathbb{R}$ are chosen so that*

$$\{z \in \mathbb{R}^d \,|\, z^T P z + 2q^T z + r \le 1\}$$

is an inscribed ellipsoid of set $\mathcal{S}$, then we have

$$1 - E[z^T P z + 2q^T z + r] \le \mathbf{P}(z \in \mathcal{S})$$

Proof: Let $f(z) = z^T P z + 2qz + r$, then $f(z) \ge 0$ for $z \in \mathcal{S}$ and $f(z) \ge 1$ for any $z \in \mathcal{S}^c$, where $\mathcal{S}^c$ is the complement of $\mathcal{S}$. Let $I_{\mathcal{S}^c}(.)$ be the indicator function on $\mathcal{S}^c$, then

$$f(z) \ge I_{\mathcal{S}^c}(z)$$

Therefore,

$$E[f(z)] \ge E[I_{\mathcal{S}^c}(z)] = \mathbf{P}(z \in \mathcal{S}^c)$$

which is the same as

$$1 - E[z^T P z + 2qz + r] \le \mathbf{P}(z \in \mathcal{S})$$

■

Note that

$$E[z^T P z + 2qz + r] = \mathbf{Tr}(PE[zz^T]) + 2q^T E[z] + r$$

The underlying probability distribution of z does not affect the nature of the bound. In other words, the bound is valid for any distribution, or an ambiguous distribution (see [10] and [16] for example), as long as its first and second moment coincide with the given ones.

Theorem 1 gives a lower bound on the probability that z falls into $\mathcal{S}$, or an upper bound on the probability that z falls outside. Now we consider how to use it to approximate the probabilistic constraint. Observe that the constraint on the system states

$$T\mathcal{X} \le b$$

can be rewritten as

$$T \begin{bmatrix} \mathbb{H} & \mathbb{G} \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{W} \end{bmatrix} \le b - T\mathbb{F}x_0$$

Let

$$z = \begin{bmatrix} \mathcal{U} \\ \mathcal{W} \end{bmatrix}, \quad \widehat{b} = b - T\mathbb{F}x_0$$

and

$$\widehat{T} = T \begin{bmatrix} \mathbb{H} & \mathbb{G} \end{bmatrix}$$

The state constraint simply becomes

$$\widehat{T}z \le \widehat{b} \tag{2}$$

Now we can then substitute the probabilistic constraint with

$$E[z^T P z + 2q^T z + r] \le 1 - \alpha$$

where P, q and r are the parameters for an inscribed ellipsoid of the convex set defined by (2).

Theorem 1 does not mention the way for selecting the inscribed ellipsoid. Of course there are many possible ways to choose the ellipsoid and as one can expect, the quality of the bound largely relies on the choice of the ellipsoid. It is not clear how to optimally choose one, since the ellipsoid that provides the sharpest bound will be dependent upon the first and second moments of z. One reasonable choice is to use the maximum volume inscribed ellipsoid (see [14]). This ellipsoid can be easily computed using conventional semidefinite programming (SDP) solvers.

Let $\mathcal{S}$ be the polyhedron defined by the linear inequalities:

$$\{\widehat{T_i}z \le \widehat{b_i},\ i = 1, ..., m.\} \tag{3}$$

The maximum volume inscribed ellipsoid of $\mathcal{S}$ is given by

$$\{Bu + d \ \mid \|u\|_2 \le 1\}$$

where $B \in \mathbb{S}^d$, and $d \in \mathbb{R}^d$ are obtained from the following log-det program:

$$\begin{aligned} &\text{maximize:} \quad \log\det B \\ &\text{subject to:} \quad \|B\widehat{T_i}^T\|_2 + \widehat{T_i}d \le \widehat{b_i} \text{ for } i = 1, ..., m \end{aligned} \tag{4}$$

Using the affine mapping $z = Bu + d$, we can obtain the formulation of the maximum volume inscribed ellipsoid in terms of z.

$$\{z \mid z^T P z + 2q^T z + r \le 1\}$$

where the transformation is given by

$$P = (BB^T)^{-1}, q = -Pd \text{ and } r = d^T P d \tag{5}$$

The approach to replace the probabilistic constraint has been shown. One issue is that originally $\widehat{T}z \le \widehat{b}$ may be unbounded, so the maximum volume inscribed ellipsoid is not applicable. This issue can be addressed by adding box constraints to the control and the disturbances.

$$-M \le \mathcal{U} \le M, -M \le \mathcal{W} \le M$$

where M is sufficiently large.

IV. Affine closed-loop controller design

We consider the following closed-loop control law

$$u_i = \bar{u}_i + \sum_{j=0}^{i-1} K_{(i,j)} w_j,\ \ i = 0, 1, ..., N-1.$$

where $K_{(i,j)}$ are constant gain matrices. This approach is similar in spirit to [19]. Also, note that this control law is an affine function of past disturbances instead of past states. Given the structure of the $\mathbb{G}$, $\mathbb{H}$, and $\mathbb{K}$ matrices in our problem, it is always possible to recover the disturbances $w_0, \ldots, w_{t-1}$ from the states $x_0, \ldots, x_t$. So, this control law can be equivalently implemented as a state feedback control law.

This control consists of a constant component and a linear combination of the uncertainties. To be consistent, it is more convenient to write them in the following form

$$\mathcal{U} = \overline{\mathcal{U}} + \mathbb{K}\mathcal{W}$$

where $\mathbb{K}$ is the gain matrix given by

$$\mathbb{K} = \begin{bmatrix} 0 & & & \\ K_{(1,0)} & 0 & & \\ K_{(2,0)} & K_{(2,1)} & & \\ \vdots & \ddots & \ddots & \\ K_{(N-1,0)} & \ldots & K_{(N-1,N-2)} & 0 \end{bmatrix}$$

As it can be seen, the gain matrix $\mathbb{K}$ is strictly block lower triangular due to the causality of the control law. We leave $\mathbb{K}$ to be a variable of the optimization problem.

To summarize, recall that our aim is to minimize

$$E[\mathcal{X}^T \mathbb{Q} \mathcal{X} + \mathcal{U}^T \mathbb{R} \mathcal{U}],$$

where

$$\begin{aligned} &\mathcal{X} = \mathbb{F}x_0 + \mathbb{H}\overline{\mathcal{U}} + (\mathbb{G} + \mathbb{H}\mathbb{K})\mathcal{W}, \\ &\mathcal{U} = \overline{\mathcal{U}} + \mathbb{K}\mathcal{W}, \\ &z = \begin{bmatrix} \mathcal{U} \\ \mathcal{W} \end{bmatrix}, \\ &\mathbf{Tr}(PE[z^T z]) + 2q^T E[z] + r \le 1 - \alpha, \end{aligned}$$

and the optimization variables are the vector $\overline{\mathcal{U}}$ and the strictly block lower triangular matrix $\mathbb{K}$. Directly in terms of these optimization variables we can write this problem as **(P2)**:

$$\begin{aligned}
\text{minimize:} \quad & \begin{bmatrix} x_0 \\ \overline{\mathcal{U}} \end{bmatrix}^T \begin{bmatrix} \mathbb{F}^T\mathbb{Q}\mathbb{F} & \mathbb{F}^T\mathbb{Q}\mathbb{H} \\ \mathbb{H}^T\mathbb{Q}\mathbb{F} & \mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R} \end{bmatrix} \begin{bmatrix} x_0 \\ \overline{\mathcal{U}} \end{bmatrix} \\
& +\mathbf{Tr}(\mathbb{K}^T(\mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R})\mathbb{K}\Sigma + 2\mathbb{K}^T\mathbb{H}^T\mathbb{Q}\mathbb{G}\Sigma) \\
\text{subject to:} \quad & \overline{\mathcal{U}}^T P_{11}\overline{\mathcal{U}} + 2q_1^T\overline{\mathcal{U}} \\
& +\mathbf{Tr}(\mathbb{K}^T P_{11}\mathbb{K}\Sigma + 2\mathbb{K}^T P_{12}\Sigma) \\
& \leq 1-\alpha-r-\mathbf{Tr}(P_{22}\Sigma)
\end{aligned}$$

Here we have partitioned P and q as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

If the problem is solved then we obtained the optimal control policy expressed as a constant term plus a linear term associated with the disturbances. The problem **(P2)** turns out to be a convex quadratic program with respect to $(\overline{\mathcal{U}}, \mathbb{K})$.

Theorem 2. *(P2) is a convex quadratic program.*

Proof: First we look at the objective. The objective is expressed as a sum of quadratic two terms, one exclusively in terms of the variable $\overline{\mathcal{U}}$ and the other exclusively in terms of the variable $\mathbb{K}$. Since $\mathbb{Q}$ and $\mathbb{R}$ are both positive semidefinite, the term in $\overline{\mathcal{U}}$ is convex. The term in $\mathbb{K}$ is convex if $\mathbf{Tr}(\mathbb{K}^T(\mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R})\mathbb{K}\Sigma)$ is convex. Since any positive semidefinite matrix can be expressed as a finite nonnegative linear combination of matrices of the type $\beta\beta^T$ ([20]), we have

$$\begin{aligned}
&\mathbf{Tr}(\mathbb{K}^T(\mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R})\mathbb{K}\Sigma) \\
&= \mathbf{Tr}(\mathbb{K}^T(\mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R})\mathbb{K}\sum_{i=1}^{t}\beta_i{\beta_i}^T) \\
&= \sum_{i=1}^{t}(\mathbb{K}\beta_i)^T(\mathbb{H}^T\mathbb{Q}\mathbb{H}+\mathbb{R})(\mathbb{K}\beta_i)
\end{aligned}$$

which is a non-negative linear combination of convex functions of $\mathbb{K}\beta_i$. So this term is convex in $\mathbb{K}$ and therefore the objective is convex.

Now we prove the constraint is also convex. As with the objective, the left hand side of the constraint is expressed as a sum of quadratic two terms, one exclusively in terms of the variable $\overline{\mathcal{U}}$ and the other exclusively in terms of the variable $\mathbb{K}$. Since P_{11} is positive semidefinite, the term in $\overline{\mathcal{U}}$ is convex. The term in $\mathbb{K}$ is convex if $\mathbf{Tr}(\mathbb{K}^T P_{11}\mathbb{K}\Sigma)$ is convex. Using the same analysis for the objective here, we conclude that $\mathbf{Tr}(\mathbb{K}^T P_{11}\mathbb{K}\Sigma)$ is a non-negative linear combination of convex functions. Hence, the constraint is a convex quadratic constraint. ■

We showed the approach to replace the probabilistic constraint and designed a causal affine control law for the resulting problem. We further proved that the this control law can be computed by solving a convex program. For completeness, the algorithm that used to solve **(P1)** is summarized in Table I.

TABLE I

THE ALGORITHM

Step 1:	Compute $\mathbb{F}$, $\mathbb{H}$, $\mathbb{G}$, Σ, $\mathbb{Q}$ and $\mathbb{R}$.
Step 2:	Construct $\widehat{T}$, $\widehat{b}$ and the set $\mathcal{S}$ using (2).
Step 3:	Solve the maximum volume inscribed ellipsoid problem (4) and obtain B and d.
Step 4:	Use the transformation (5) to get P, q and r.
Step 5:	Solve **(P2)** for $\overline{\mathcal{U}}$ and $\mathbb{K}$.

V. NUMERICAL EXAMPLES

In this section we choose two state-constrained stochastic LQR problems with 2 states and 2 control inputs, showing the reliability of the approach. We compare our approach with the certainty equivalent approach only, which replaces random variables with their expected values. As we mentioned before, there exist approximation methods for scalar cases. For the Bernstein approximation method, there is no immediate result for multi-dimensional cases. We also tried the scenario approximation method, however, the number of the introduced constraints that replace the probabilistic constraint easily grows to thousands and the solver time is too long to be practical. All the examples are implemented in Matlab, using Yalmip [18] and SDPT3 [17].

The system dynamics and disturbance parameters are given by

$$A = \begin{bmatrix} 1.02 & -0.1 \\ 0.1 & 0.98 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0.05 & 0.5 \end{bmatrix},$$

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, Q_N = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}$$

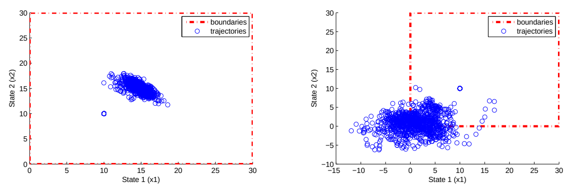
The initial system state and the uniform box state constraint are given by

$$x_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq x_k \leq \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

The disturbance vectors are multi-dimensional normal and i.i.d..

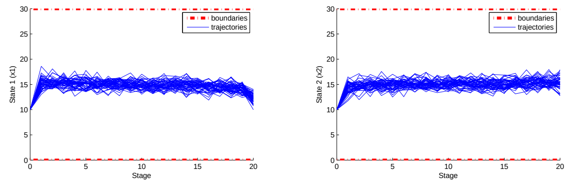
$$w_k \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.81 & -0.648 \\ -0.648 & 0.81 \end{bmatrix}\right)$$

The probability requirement α is set to be 0.8. The trajectories of our model and the certainty equivalent model are shown in the following figures.

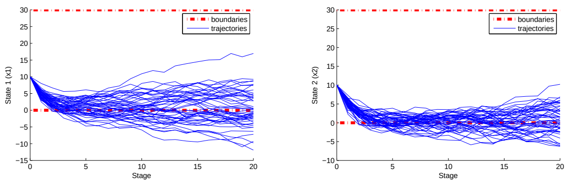


(a) Chebyshev bound (b) Certainty equivalent

Fig. 1. 2D trajectories for the 20-stage case



(a) Chebyshev bound: x1 (b) Chebyshev bound: x2



(c) Certainty equivalent: x1 (d) Certainty equivalent: x2

Fig. 2. state trajectories for the 20-stage case

First we test our approach on problems with box state constraints. Figure 1(a) contains the trajectories of the state-constrained stochastic LQR problem with a horizon containing 20 periods solved using our approach. Figure 1(b) is those solved using the certainty equivalent model. The simulation is repeated 50 times. As we can see, our model, which uses Chebyshev bound successfully keeps the trajectories within the red dotted boundary box, which represents the state constraints. The trajectories of the certainty equivalent fall out of the boundaries quickly as the controller acts risky towards the origin, where the cost is minimized. Figure 2 gives the trajectories in detail by showing them state-by-state. Figure 2(a) and Figure 2(b) are the state trajectories for our model and Figure 2(c) and Figure 2(d) are those for the certainty equivalent model.

We also test our approach on a 10-stage tracking problem. The trajectory we are trying to track is given by

$$y_k = \begin{bmatrix} 5\sin(0.2k) \\ 5\cos(0.2k) \end{bmatrix}$$

The feasible trajectory region is defined by manipulating the upper and lower bounds of the states

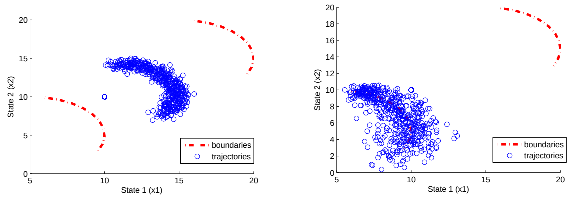
$$y_k - 5 \le x_k \le y_k + 5$$

The distribution of the i.i.d. disturbances is different in this example

$$w_k \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.12 & 0 \\ 0 & 0.12 \end{bmatrix}\right)$$

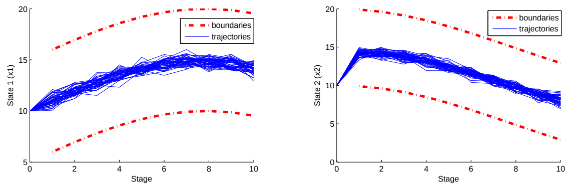
The simulation results are shown below.

Figure 3(a) shows the trajectories of the 10-stage tracking problem using our approach and Figure 3(b) shows those solved using the certainty equivalent model. Figure 4(a) and Figure 4(b) are the state trajectories for our model and Figure 4(c) and Figure 4(d) are those for the certainty equivalent model. This example gives an alternative way to formulate the tracking problem other than using penalties. We can see that

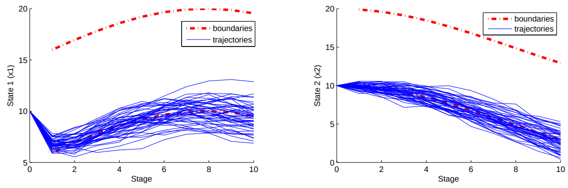


(a) Chebyshev bound (b) Certainty equivalent

Fig. 3. 2D trajectories for the 10-stage tracking case



(a) Chebyshev bound: x1 (b) Chebyshev bound: x2



(c) Certainty equivalent: x1 (d) Certainty equivalent: x2

Fig. 4. State trajectories for the 10-stage tracking case

our approach performs robustly against the uncertainty and stays in the desired trajectory region (between the red dotted curves) while the certainty equivalent approach disregards the risk of violating the constraints and behaves very aggressively.

VI. Conclusion and Future Work

In this paper, we proposed an approach to approximate probabilistic constraints using the multi-dimensional Chebyshev bound and the maximum volume inscribed ellipsoid. We also designed an optimal causal affine controller for the approximate problem and proved the convexity of it. A practical algorithm is summarized to solve the problem which can be developed using general quadratic program solvers. This approach also can be used as a subroutine in a model predictive control algorithm for state-constrained stochastic LQR problems.

The results derived in this paper can also be applied to solve other problems coupled with joint probabilistic constraints. For example, a noisy input can be modeled using joint probabilistic constraints, which can be replaced by a convex deterministic one with our approximation approach. Also, the causal affine controller can be extended to nonlinear controller if one can choose qualified basis functions of the disturbances to replace the affine disturbance structure.

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